From Lie groupoids to resolutions of singularities. Applications to symplectic and Poisson resolutions.

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Abstract

We use the techniques of integration of Poisson manifolds into symplectic Lie groupoids to build symplectic resolutions (=desingularizations) of the closure of a symplectic leaf and characterise the resolutions obtained by this procedure. More generally, we show how Lie groupoids can be used to lift singularities, in particular when one imposes a compatibility condition with an additional structure given by a multi-vector field.

Contents

1 Introduction

1.1 Presentation

In the literature, the idea of replacing a Poisson variety X and its oddities by a symplectic manifold Z that projects on X through a Poisson map appears in two a priori different contexts, depending on what is singular: the variety or the Poisson structure. In algebraic geometry, see [1, 10, 12], given a variety W endowed with a Poisson structure of maximal rank (= symplectic) at regular points, Beauville has introduced the notion of *symplectic resolution*, which consists of a resolution $\Sigma \xrightarrow{\phi} W$ in the sense of Hironaka's Big Theorem with Σ symplectic and ϕ a Poisson map. In differential geometry, see [2, 7, 8], a Poisson manifold M is being replaced by a manifold Γ, called *symplectic groupoid*, which has a (maybe local) Lie groupoid structure over M, together with some compatible symplectic structure, the Poisson map being then simply either the source map or the target map.

This article discusses the possibility to go from symplectic groupoids to symplectic resolutions, and conversely. To make a long story short, *our aim is to show how the symplectic groupoid of a Poisson manifold can be used to desingularize the closure of a symplectic leaf.* A secondary aim is to use Poisson groupoids (more generally Lie groupoids endowed with multiplicative k-vector fields) to find Poisson resolutions (more generally resolutions compatible with a k-vector field) of the closure of an algebroid leaf.

We point out the most crucial differences that exist between the theories of symplectic resolutions and symplectic groupoids, and explain briefly how we avoid or unify them.

1. First of all, symplectic resolutions belong to the world of algebraic geometry while the theory of symplectic groupoids has been developed on real smooth manifolds mainly (but most of its results extend to the holomorphic setting, see [17]). To avoid this difficulty, one possibility could be to rewrite the theory of symplectic groupoids in the language of algebraic geometry. We make the opposite choice, and decide work inside the world of differential geometry.

2. More precisely, we mimic, within differential geometry, the definition of symplectic resolutions introduced by Beauville. The object that we are going to try to desingularize is the closure $\overline{\mathcal{S}}$ of a locally closed symplectic leaf $\mathcal{S}$ of a Poisson manifold (M, π). This closure $\overline{\mathcal{S}}$ behaves precisely as in the algebraic case since
   (a) regular points (= points in $\mathcal{S}$) form a dense open subset of $\overline{\mathcal{S}}$, (as the regular part of an algebraic variety does) and
   (b) since $\mathcal{S}$ is a symplectic leaf, the restriction to $\overline{\mathcal{S}}$ of the Poisson structure is symplectic at regular points (= points in $\mathcal{S}$).
3. Symplectic resolutions and symplectic groupoids behave differently with respect to dimensions. On the one hand, a symplectic resolution of a Poisson variety of dimension k has the same dimension k. On the other hand, the symplectic groupoid procedure doubles the dimension, id. est, it starts from a Poisson manifold (M, π) of dimension n and builds a symplectic Lie groupoid of dimension $2n$. In particular, the symplectic groupoid itself can by no way be itself the symplectic resolution. But the main idea of this paper is that the symplectic groupoid $\Gamma \rightrightarrows M$ can give a symplectic resolution of the closure $\overline{\mathcal{S}}$ of a symplectic leaf $\mathcal{S}$ by going through the following steps:
   (a) we choose carefully some submanifold L of M, included into $\overline{\mathcal{S}}$, and whose intersection with $\mathcal{S}$ is Lagrangian in $\mathcal{S}$, and
   (b) we apply a procedure called symplectic reduction to the submanifold $\Gamma_L = s^{-1}(L)$ (where $s : \Gamma \to M$ is the source map) which is coisotropic in the symplectic manifold Γ. This symplectic reduction reduces the dimension, and involves in this case a strong Lie groupoid machinery, and
   (c) we obtain (under some conditions) a symplectic manifold that gives a symplectic resolution of $\overline{\mathcal{S}}$. The Poisson map onto $\overline{\mathcal{S}}$ is induced by the target map of the groupoid $\Gamma \rightrightarrows M$.

Let us point out two advantages of this method. First, it unifies two theories of desingularization. More precisely, $\overline{\mathcal{S}}$ may be "singular" in two ways: it can be a singular variety (whatever it means in the smooth/holomorphic context), in this case, what is "desingularized" is a singularity of the variety. But $\overline{\mathcal{S}}$ can very well be a smooth manifold of dimension $2l$, but the Poisson structure is singular (= of rank $< 2l$) at singular points (= points in $\overline{\mathcal{S}} \backslash \mathcal{S}$). In this case, what is "desingularized" is a singularity of the Poisson structure.

Second, it generalises to other types of desingularizations compatible with some additional geometrical structure given by a multi-vector field: Poisson resolutions in particular, but also contact resolutions and twisted symplectic resolutions (to be studied in an other work), to mention just the main ones. In order to reach that level of generality, we shall work, as much as we can, in the very general case of a Lie groupoid endowed with a multiplicative k-vector field.

The organisation of the paper is as follows. In Section 2, we ask the following *question*: "Given a Lie groupoid, how can one desingularize the closure $\overline{\mathcal{S}}$ of an algebroid leaf $\mathcal{S}$?" and our (partial) *answer* is: "With the help of an algebroid crossing (see Definition 2.7), under some integrability assumptions".

In Section 3, we recall from [15] the following *fact*: "a multiplicative k-vector field on a groupoid induces a k-vector field π_M on the algebroid leaves". Then we raise the *question*:

“Given an algebroid and a multiplicative k-vector field on the corresponding groupoid (with $k \geq 2$), how can one desingularize $\overline{\mathcal{S}}$ in a way that is compatible with this k-vector field ?” and we suggest the following *answer*: “With the help of a algebroid crossing coisotropic with respect to π_M, under some integrability condition”.

In Section 4, we raise the *question*: “How can we construct a symplectic resolution of the closure $\overline{\mathcal{S}}$ of a symplectic leaf $\mathcal{S}$ of a Poisson manifold ?” and we propose the following *answer*: “ Our previous results, applied to the special case of a Lagrangian crossing (see Definition 4.4) give automatically a symplectic resolution”.

In Section 5, we give a characterisation of symplectic resolutions of the previous forms in terms of compatibility with respect to a Lagrangian crossing.

We then present in Section 6 examples of such resolutions. Section 6.1 presents a trivial example that illustrates in a very clear way our results: we lift, in the world of real geometry, the singularity at the origin of the real Poisson bracket on $\mathbb{R}^2$ given by

$$\{x, y\} = x^2 + y^2.$$

In Section 6.2, the celebrated Springer resolution is rediscovered as a particular case of the previous constructions. Note that the second of these examples lifts a singularity of the variety, while the first one lifts a singularity of the Poisson structure. We then present an example of Poisson resolution, namely the Grothendieck resolution endowed with its Evens-Lu Poisson structure. We finish with a discussion of a famous symplectic resolution: the minimal resolution of $\mathbb{C}^2/G$ with $G = \frac{\mathbb{Z}}{l\mathbb{Z}}$.

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1.3 Notations and basic facts about Lie groupoids

For self-containess of this paper, we recall several facts about Poisson manifolds, Lie algebroids, and Lie groupoids. We recall in particular how to adapt to the holomorphic setting

the theories of symplectic groupoids and holomorphic Poisson structures, an adaptation systematically studied in [17].

Complex and real manifolds. We try to state and prove results which are valid in both complex and real differential geometries. By a *manifold*, we mean *a complex manifold or a real (smooth) manifold*. Then words like functions, vector fields, vector bundles, sections should be understood as being holomorphic or smooth, depending on the context. Moreover, in complex geometry, we have to work with local functions, local sections or local vector fields rather than their global counterparts. Given a manifold M and a vector bundle $A \to M$, we denote simply by $\mathcal{F}(M)$ and $\Gamma(A \to M)$ the sheaf of local functions and local sections respectively. Sections over an open subset V are denoted by $\Gamma(A_{|_V} \to V)$. When we write identities of the form $f = g$ for local functions f and g, it should be understood that the identity takes place on the intersection of their domain of definition. When we say that a local function f vanishes on a submanifold N, we mean of course that f vanishes on the intersection of N with the domain of definition of f. Similarly, terms of the form $X[f_1 \cdots, f_k]$, where X is a a local k-vector field, and $f_1, \cdots, f_k$ are local functions are to be considered only where it is defined. All these slight abuses of terminology shall be systematically omitted in the text.

All our manifolds are supposed to be Hausdorff. Since this last point can be ambiguous while speaking about Lie groupoid, we sometime say Hausdorff Lie groupoid to emphasise on this assumption.

Fibered product, definition and convention. Given a triple M_1, M_2, N of manifolds and maps $\phi_1 : M_1 \to N$ and $\phi_2 : M_2 \to N$, we denote by $M_1 \times_{\phi_1, N, \phi_2} M_2$ the fibered product:

$$M_1 \times_{\phi_1, N, \phi_2} M_2 := \{(m_1, m_2) \in M_1 \times M_2 | \phi_1(m_1) = \phi_2(m_2)\}.$$

This fibered product is itself a manifold as soon as one of the maps ϕ_1, ϕ_2 is a submersion.

Multi-vector fields, definition and convention. A k-vector field π is a section of $\wedge^k TM \to M$ (the wedge product being over $\mathbb{R}$ or $\mathbb{C}$ depending on the context). Throughout this paper, we will almost never have to consider two different k-vector fields defined on the same space. It is therefore very convenient for us to always denote by π_M the k-vector field over M that we consider. Recall that a k-vector field π_M defines a skew-symmetric k-derivation of the algebra $\mathcal{F}(U)$ of functions on an open subset $U \subset M$ with the help of the skew-symmetric k-derivation

$$F_1, \cdots, F_k \to (x \to (\pi_M)_{|_x}[\mathrm{d}_x F_1, \cdots, \mathrm{d}_x F_k])$$

We denote by $\{\cdot, \cdots, \cdot\}_M$ the previous skew-symmetric k-derivation. We denote by $[\![\cdot, \cdot]\!]_{TM}$ the Schouten bracket of multivector fields.

Holomorphic/real Lie algebroids, definition. With some care, we define smooth and holomorphic algebroids.

Definition 1.1. *A* real Lie algebroid *A over a smooth manifold M is a smooth vector bundle $A \to M$ together with a smooth collection, for each $x \in M$, of linear maps $\rho : A_x \to$*

T_xM *called the anchor, and an* $\mathbb{R}$*-linear skew-symmetric bracket* $[\cdot,\cdot] : \Gamma A \times \Gamma A \to \Gamma A$ *on the space of local smooth sections which satisfies the Jacobi identity, and such that*

(1) $[X, fY] = f[X,Y] + \rho(X)(f)Y$

(2) $\rho([X,Y]) = [\rho(X), \rho(Y)]$

We denote by $(A \to M, \rho, [\cdot,\cdot])$ *a Lie algebroid.*

A holomorphic Lie algebroid *is a smooth Lie algebroid* $(A \to M, \rho, [\cdot,\cdot])$ *over a complex manifold* M*, where* $A \to M$ *is a complex vector bundle, such that the anchor map is holomorphic, and the bracket* $[\cdot,\cdot]$ *restricts to a* $\mathbb{C}$*-linear Lie algebra structure on holomorphic sections over any open subset* $U \subset M$*.*

We warn the reader that holomorphic Lie algebroid should not be confused with Complex Lie algebroid (CLA) in the sense of [24].

The foliation of a Lie algebroid. The distribution $\coprod_{x\in M} \rho(A_x)$ is integrable, though its rank is not constant. Moreover, the leaves, called *algebroid leaves*, are smooth immersed submanifolds, which are immersed complex submanifolds if the Lie algebroid is a holomorphic Lie algebroid, see [17].

k-differential. Recall from [25] that $\Gamma(\wedge^\bullet A \to M)$ is a (sheaf of) Gerstenhaber algebra when equipped with the wedge product and the unique natural natural extension $[\![\cdot,\cdot]\!]_A$ of the bracket of sections of $A \to M$ (extension which is again called Schouten bracket). A *k-differential* (see Section 2 in [15]) is a linear operator $\delta : \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^{\bullet+k-1}A)$ satisfying

$$\begin{cases} \delta(P \wedge Q) = (\delta P) \wedge Q + (-1)^{p(k+1)} P \wedge \delta Q, \\ \delta\left([\![P,Q]\!]_A\right) = [\![\delta P, Q]\!]_A + (-1)^{(p+1)(k+1)} [\![P, \delta Q]\!]_A. \end{cases} \tag{1}$$

Holomorphic/real Lie groupoids, definition. There is no issue in defining Lie groupoids in both the real on the complex case, see [17]. We could say that a holomorphic/real Lie groupoid is a small category where objects, arrows and operations are holomorphic/smooth. We prefer to give a long but down-to-the-earth definition.

Definition 1.2. *A* holomorphic/real Lie groupoid $\Gamma \rightrightarrows M$ *consists of two complex/real manifolds* Γ *and* M*, together with two holomorphic/smooth surjective submersions* $s, t : \Gamma \to M$*, called the source and the target maps, and a holomorphic/smooth inclusion* $\varepsilon : M \to \Gamma$*, which admits a group law, that it to say, such that there exists* (i) *a holomorphic/smooth inverse map* $\Gamma \to \Gamma$ *denoted* $\gamma \to \gamma^{-1}$ *permuting the source and target maps and* (ii) *a holomorphic/smooth map* $\Gamma \times_{t,M,s} \Gamma \to \Gamma$ *denoted* $(\gamma_1, \gamma_2) \to \gamma_1 \cdot \gamma_2$ *satisfying*

$$\begin{array}{rcll} s(\gamma_1 \cdot \gamma_2) & = & s(\gamma_1) & \forall (\gamma_1, \gamma_2) \in \Gamma \times_{t,M,s} \Gamma \\ t(\gamma_1 \cdot \gamma_2) & = & t(\gamma_2) & \forall (\gamma_1, \gamma_2) \in \Gamma \times_{t,M,s} \Gamma \\ (\gamma_1 \cdot \gamma_2) \cdot \gamma_3 & = & \gamma_1 \cdot (\gamma_2 \cdot \gamma_3) & \forall (\gamma_1, \gamma_2, \gamma_3) \in \Gamma \times_{t,M,s} \Gamma \times_{t,M,s} \Gamma \\ \varepsilon(s(\gamma)) \cdot \gamma & = & \gamma & \forall \gamma \in \Gamma \\ \gamma \cdot \varepsilon(t(\gamma)) & = & \gamma & \forall \gamma \in \Gamma \\ \gamma \cdot \gamma^{-1} & = & \varepsilon(s(\gamma)) & \forall \gamma \in \Gamma \\ \gamma^{-1} \cdot \gamma & = & \varepsilon(t(\gamma)) & \forall \gamma \in \Gamma \\ (\gamma_1 \cdot \gamma_2)^{-1} & = & \gamma_2^{-1} \cdot \gamma_1^{-1} & \forall (\gamma_1, \gamma_2 \in \Gamma) \in \Gamma \times_{t,M,s} \Gamma \end{array}$$

Of course, the previous list of requirements is redundant, and several of the previous axioms could be erased.

To any holomorphic/real Lie groupoid, there is an holomorphic/real algebroid associated with, (see [19] Section 3.5 for the real case, the complex case is similar). This Lie algebroid is defined by the vector bundle $A := \coprod_{x\in M} A_x$ where $A_x = \ker(\mathrm{d}_x s) \subset T_{\varepsilon(x)}\Gamma$. The converse is not true: a Lie algebroid may not be the Lie algebroid of a Lie groupoid, see [3]. When such a Lie groupoid exists, we say that the Lie groupoid is question *integrates* the algebroid. Recall from [17] that, when a holomorphic Lie algebroid, when considered as a real Lie algebroid by forgetting the complex structure, integrates to a smooth source-simply-connected Lie groupoid $\Gamma \rightrightarrows M$, then $\Gamma \rightrightarrows M$ inherits an unique natural complex structure that turns it into a holomorphic Lie groupoid.

We use the following notations. For any $L, L' \subset M$, one introduces $\Gamma_L := s^{-1}(L)$, $\Gamma^{L'} := t^{-1}(L')$ and $\Gamma_L^{L'} = \Gamma_L \cap \Gamma^{L'}$. Given a point $x \in M$, we use the shorthands Γ_x, Γ^x for $\Gamma_{\{x\}}$ and $\Gamma^{\{x\}}$ respectively.

Left- and right-invariant vector fields. When the Lie algebroid integrates to a Lie groupoid, sections of $A \to M$ are in one-to-one correspondence with right-invariant (resp. left-invariant) vector fields on $\Gamma \rightrightarrows M$. Given a section $X \in \Gamma(A)$, we denote by $\overrightarrow{X}$ (resp. $\overleftarrow{X}$) the right-invariant (resp. left) vector fields corresponding to it.

We extend this convention to sections of $\wedge^\bullet A$. Namely, following [15], we define, for all $k \geq 1$ and all (maybe local) sections $e_1, \ldots, e_k \in \Gamma(A \to M)$, a right-invariant k-vector field on $\Gamma \rightrightarrows M$, by $\overrightarrow{e_1 \wedge \cdots \wedge e_k} = \overrightarrow{e_1} \wedge \cdots \wedge \overrightarrow{e_k}$ (resp. $\overleftarrow{e_1 \wedge \cdots \wedge e_k} = \overleftarrow{e_1} \wedge \cdots \wedge \overleftarrow{e_k}$) and extend this correspondence by multilinearity. For $k = 0$, sections of $\wedge^0 A$ are simply functions: we define then $\overrightarrow{f} = s^* f$ (resp. $\overleftarrow{X} = t^* f$).

The map from $\Gamma(\wedge^\bullet A \to M) \to \mathcal{X}^\bullet(\Gamma)$ given by $X \to \overrightarrow{X}$ is a morphism of Gerstenhaber algebras, where $\mathcal{X}^\bullet(\Gamma)$ stands for the Gerstenhaber algebra of multivector fields. When dealing with left-invariant vector fields, the same result holds up to a sign, namely the map defined by $X \to (-1)^k \overleftarrow{X}$ for all $X \in \Gamma(\wedge^k A \to M)$ is a morphism of Gerstenhaber algebras.

Modules of a Lie groupoid. A left action of a Lie groupoid $\Gamma \rightrightarrows M$ on a pair (X, ϕ), with X a manifold and $\phi : X \to M$, is a holomorphic/smooth map from $\Gamma \times_{t,M,\phi} X$ (which is a manifold because the target map is a submersion) to X, called the *action map* and denoted by $(\gamma, x) \to \gamma \cdot x$, which satisfies the following axioms (see [19] Section 1.6):

$$\begin{cases} \gamma \cdot x & \in & \phi^{-1}(s(x)) & \forall (\gamma, x) \in \Gamma \times_{x,M,\phi} X \\ \varepsilon(\phi(x)) \cdot x & = & x & \forall x \in X \\ \gamma_1 \cdot (\gamma_2 \cdot x) & = & (\gamma_1 \cdot \gamma_2) \cdot x & \forall (\gamma_1, \gamma_2, x) \in \Gamma \times_{t,M,s} \Gamma \times_{t,M,\phi} X \end{cases}$$

Right action are defined in the same way. We say then that (X, ϕ) is a *(left or right) Γ-module*. Given a left (resp. right) action of $\Gamma \rightrightarrows M$ on $X \to M$, we call *quotient space* and denote by $\Gamma \backslash X$ (resp. X/Γ) the space of orbits of the action, id est, the space $X/\sim$ where $\sim$ is the equivalence relation that identifies x and $\gamma \cdot x$ (resp. $x \cdot \gamma$) for any $(\gamma, x) \in \Gamma \times_{t,M,\phi} X$ (resp. $X \times_{\phi,M,s} \Gamma$).

A groupoid $\Gamma \rightrightarrows M$ acts on (M, Id) by

$$\gamma \cdot m = t(\gamma) \text{ if } s(\gamma) = m.$$

The following result concatenates several results in [19].

Lemma 1.3. *Let $\Gamma \rightrightarrows M$ be a source-connected Lie groupoid integrating $A \to M$. Any two points in M are is the same Lie algebroid leaf if and only if they are in the same orbit $\mathcal{S}$ of the action of $\Gamma \rightrightarrows M$ on M (i.e. if and only if they are source and target of some $\gamma \in \Gamma$). Moreover, the map $\gamma \to (s(\gamma), t(\gamma))$ restricts to a submersion from $\Gamma_{\mathcal{S}} = \Gamma_{\mathcal{S}}^{\mathcal{S}} = \Gamma^{\mathcal{S}}$ to $\mathcal{S} \times \mathcal{S}$.*

We say that a subset $X \subset M$ is Γ-*connected* if and only if, for any two $x, y \in X$ there exists a finite sequence $x = x_0, x_1, \cdots, x_l = y$ such that, for all $i \in \{0, \cdots, n_1\}$, either x_i, x_{i+1} are the same connected component of X, or x_i and x_{i+1} are the source and target of some $\gamma \in \Gamma$.

Holomorphic Poisson manifold and integration. We recall that for any Poisson manifold (M, π_M), the cotangent bundle $T^*M \to M$ is endowed with a natural Lie algebroid structure $(T^*M, \pi_M^\#, [\cdot,\cdot]_{\pi_M})$.

A *real/holomorphic Poisson manifold* (M, π_M) is a complex manifold M endowed with a smooth/holomorphic bivector field π_M such that $[\![\pi_M, \pi_M]\!]_{TM} = 0$. Recall from [17] that the real part $\mathfrak{Re}(\pi_M)$ and the imaginary part $\mathfrak{Im}(\pi_M)$ of π_M are compatible smooth Poisson structures.

We refer to [2] for an overview of the theory of integration of Poisson manifolds to symplectic groupoid in the real case, and we give a brief overview. A *symplectic Lie groupoid* is a pair $(\Gamma \rightrightarrows M, \omega_\Gamma)$ where $\Gamma \rightrightarrows M$ is a Lie groupoid and ω_Γ a symplectic structure on Γ which is compatible with respect to the multiplication. A symplectic Lie groupoid $(\Gamma \rightrightarrows M, \omega_\Gamma)$ is said to *integrate* (M, π_M) if $s_*\pi_\Gamma = \pi_M$. Any source-simply-connected Lie groupoid $\Gamma \rightrightarrows M$ integrating the algebroid $(T^*M, \pi_M^\#, [\cdot,\cdot]^{\pi_M},)$ admits a unique symplectic structure integrating (M, π_M).

Assume that there exists a source-simply-connected symplectic Lie groupoid $\Gamma \rightrightarrows M$ integrating the algebroid $(T^*M, (\mathfrak{Re}(\pi_M))^\#, [\cdot,\cdot]_{\mathfrak{Re}(\pi_M)})$. According to [17], the Lie groupoid $\Gamma \rightrightarrows M$ is indeed a holomorphic symplectic Lie groupoid with respect to holomorphic symplectic form ω whose real part is $\frac{1}{4}\omega_R$.

2 Resolution of the closure of an algebroid leaf.

2.1 Definition of a resolution in real and complex geometries

In the context of algebraic geometry, a *resolution* of a variety W is pair (Z, ϕ) where Z is a smooth (= without singularities) variety and $\phi : Z \to W$ is a proper regular birational map onto W. In particular, the restriction of ϕ to $\phi^{-1}(W_{reg}) \to W_{reg}$ is biregular (where W_{reg} stands for the regular part of W).

One difficulty arises when we try to reformulate this definition in the context of complex or real geometry: what kind of “singular” varieties should we consider ? We avoid this

difficulty by restricting ourself to the following situation. All the singular smooth/complex "varieties" that we are going to study are of the form $\bar{\mathcal{S}}$ where $\mathcal{S}$ is a (not closed in general) locally closed (= embedded) submanifold of a manifold M, and where the closure is with respect to the usual topology of M.

In this context, by a resolution, we mean the following:

Definition 2.1. *Let $\bar{\mathcal{S}}$ be a the closure of an locally closed (= embedded) submanifold $\mathcal{S}$ of a complex/real manifold M. A* resolution *of $\bar{\mathcal{S}}$ is a pair (Z, ϕ) where Z is a complex/real manifold and $\phi : Z \to M$ is a holomorphic/smooth map such that*

1. *$\phi(Z) = \bar{\mathcal{S}}$,*
2. *$\phi^{-1}(\mathcal{S})$ is dense in Z,*
3. *the restricted map $\phi : \phi^{-1}(\mathcal{S}) \to \mathcal{S}$ is an biholomorphism/diffeomorphism.*

When $\phi : \phi^{-1}(\mathcal{S}) \to \mathcal{S}$ is only an étale map (i.e. a surjective local biholomorphism/diffeomorphism), then we speak of an étale resolution *of $\bar{\mathcal{S}}$. When $\phi : \phi^{-1}(\mathcal{S}) \to \mathcal{S}$ is only a covering (i.e. $\phi^{-1}(\mathcal{S})$ is a connected set and $\phi : \phi^{-1}(\mathcal{S}) \to \mathcal{S}$ is an étale map), then we speak of a* covering resolution *of $\bar{\mathcal{S}}$.*

For étale or covering resolutions, by saying that the typical fiber is isomorphic to some discrete set Z, we mean that $\phi^{-1}(x) \simeq Z$ for all $x \in \mathcal{S}$.

Remark 2.2. Let M be a nonsingular variety over $\mathbb{C}$ and $W \subset N$ an irreducible subvariety. Then the regular part W_{reg} of W is a locally closed complex submanifold of M and $\overline{W_{reg}} = W$ (where the closure is with respect to the usual topology of M). A resolution (in the sense of algebraic geometry) of W is also a holomorphic resolution (in the sense of Definition 2.1) of $\overline{W_{reg}}$.

Remark 2.3. Since $\mathcal{S}$ is dense in $\overline{\mathcal{S}}$, Condition 1 in Definition 2.1 implies Condition 2 in the complex case.

Remark 2.4. It deserves to be noted that $\bar{\mathcal{S}}$ may be itself a manifold with no singularity: take, for instance, $M = \mathbb{C}$ and $\mathcal{S} = \mathbb{C}^*$. As a consequence, in the case where $\bar{\mathcal{S}}$ happens to be an affine subvariety of $\mathbb{C}^n$, then $\mathcal{S}$, which is always included in $(\bar{\mathcal{S}})_{reg}$, may be strictly included into $(\bar{\mathcal{S}})_{reg}$. So that, in the particular case where $\bar{\mathcal{S}}$ is itself an affine variety, a holomorphic resolution (Z, ϕ) of $\bar{\mathcal{S}}$ is the sense of Definition 2.1 may not be a resolution in the sense of algebraic geometry, even when (Z, ϕ) is in the category of algebraic varieties.

A *morphism between two étale resolutions* (Z_i, ϕ_i)*,* $i = 1, 2$ is a holomorphic/smooth map $\Psi : Z_1 \to Z_2$ such that $\phi_2 \circ \Psi = \phi_1$, i.e. such that the following diagram commutes

$$\begin{tikzcd} Z_1 \arrow[r, "\Psi"] \arrow[dr, "\phi_1"'] & Z_2 \arrow[d, "\phi_2"] \\ & \bar{\mathcal{S}} \end{tikzcd}$$

2.2 Lie groupoids and resolutions of the closure of an algebroid leaf.

We explain in this section how to build a resolution of the closure $\bar{\mathcal{S}}$ of a locally closed (= embedded) submanifold $\mathcal{S}$ of a manifold M which happens to be an algebroid leaf of an integrable algebroid.

Lemma 2.5. *Let $\mathcal{S}$ be a locally closed leaf of a Lie algebroid $(A \to M, \rho, [\cdot,\cdot])$. Then $\bar{\mathcal{S}}$ is a disjoint union of algebroid leaves.*

Proof. Though one could derive this result directly from general considerations about integrable distributions, we give an easy proof that uses local integration. Let $\Gamma \rightrightarrows M$ be a source-connected local Lie groupoid integrating the algebroid $(A \to M, \rho, [\cdot,\cdot])$. Choose a point $m \in \bar{\mathcal{S}}$ and an element $\gamma \in \Gamma_m$. In any neighbourhood W of $\gamma \in \Gamma$, there exists, since the source map is a submersion, at least one element $\gamma' \in W$ which is mapped by the source map to an element in $\mathcal{S}$. The target $t(\gamma')$ of such an element belongs to $\mathcal{S}$. Since W can be chosen arbitrary small, $t(\gamma)$ belongs to $\bar{\mathcal{S}}$. By Lemma 1.3, $\bar{\mathcal{S}}$ needs to be an union of algebroid leaves. □

Remark 2.6. In view of Lemma 2.5, we may suggest that the notion of stratified space would be more relevant here and $\bar{\mathcal{S}}$ should be called a "strate", however, it seems to be customary to speak about "algebroids leaves" (and in particular of "symplectic foliation" in the case of Poisson manifold) so we adhere to this traditional vocabulary.

The construction of étale resolutions of $\bar{\mathcal{S}}$ requires the following object.

Definition 2.7. *Let $(A \to M, \rho, [\cdot,\cdot])$ be a Lie algebroid and $\mathcal{S}$ a locally closed (= embedded) algebroid leaf. We say that a submanifold L of M is an* algebroid crossing of $\bar{\mathcal{S}}$ *if and only if the following conditions are satisfied*

1. *L is a submanifold of M,*
2. *$L \subset \bar{\mathcal{S}}$,*
3. *L intersects all the algebroid leaves contained in $\bar{\mathcal{S}}$,*
4. *$L \cap \mathcal{S}$ is dense in L,*
5. *the vector bundle $B \to L \cap \mathcal{S}$ defined over $L \cap \mathcal{S}$ by*

$$B_x = \rho^{-1}(T_x L) \quad \forall x \in L \cap \mathcal{S}$$

   extends to a (holomorphic/smooth) vector bundle over L, denoted again by $B \to L$ and called normalisation of the algebroid crossing *L.*

Remark 2.8. Note that for a manifold L that satisfies Conditions (1)-(4), the normalisation, if any, is unique. note also that Condition 52) is a consequence of Condition (4).

Example 2.9. The reader may ask how it can be that $\overline{\mathcal{S}}$ is singular while L is smooth, especially taking under account that Definition 2.7(3) forces L to go through the singularities of $\overline{\mathcal{S}}$. These conditions are, however, perfectly compatible. Assume, for instance, that M is a vector space, and that $\mathcal{S}$ is a cone, minus the origin. Then $\overline{\mathcal{S}} = \mathcal{S} \cap \{0\}$ is the cone itself. Any straight line L contained in that cone is a smooth submanifold of M

and admits a non-empty intersection with both $\mathcal{S}$ and $\{0\}$. Here is an example for which this situation occurs. The Lie group $\mathrm{SO}_n(\mathbb{C})$ acts on $\mathbb{C}^n$, and one can form the action groupoid $\mathrm{SO}_n(\mathbb{C}) \times \mathbb{C}^n \rightrightarrows \mathbb{C}^n$. The following set is an algebroid leaf of the Lie algebroid of the previous groupoid:

$$\mathcal{S} = \left\{(z_1, \cdots, z_n) \in \mathbb{C}^n - \{0\} \mid z_1^2 + \cdots + z_n^2 = 0\right\}.$$

Any straight line L through $\mathcal{S}$ is an algebroid crossing with normalisation $\mathrm{Lie}(H) \times L \to L$, where H is the stabiliser of L in $\mathrm{SO}_n(\mathbb{C})$.

By density of $L \cap \mathcal{S}$ in L, the inclusion $\rho(B_x) \subset T_x L$ holds for any $x \in L$. In particular:

Lemma 2.10. *The normalisation $B \to L$ of an algebroid crossing L of $\overline{\mathcal{S}}$, admits a unique algebroid structure such that the inclusion map into $A \to M$, is an algebroid morphism. (In other words, the normalisation $B \to L$ of an algebroid crossing L is a subalgebroid of $(A \to M, \rho, [\cdot,\cdot])$).*

The following Proposition is the main result of this section. It makes it possible to construct an étale resolution of $\overline{\mathcal{S}}$ out of an algebroid crossing, and helps us to decide whether this étale resolution turns to be a covering resolution or a resolution.

Proposition 2.11. *Let $(A \to M, \rho, [\cdot,\cdot])$ be a Lie algebroid, $\mathcal{S}$ a locally closed orbit of this algebroid, and L an algebroid crossing of $\bar{\mathcal{S}}$ with normalisation $B \to L$. If*

1. *there exists a source-connected Hausdorff Lie groupoid $\Gamma \rightrightarrows M$ integrating the Lie algebroid $(A \to M, \rho, [\cdot,\cdot])$ and*
2. *there exists a sub-Lie groupoid $R \rightrightarrows L$ of $\Gamma \rightrightarrows M$, closed as a subset of Γ_L^L, integrating the subalgebroid $B \to L$,*

then

1. *$(Z(R), \phi)$ is an étale resolution of $\bar{\mathcal{S}}$, where*
   - *(a) $Z(R) = R\backslash\Gamma_L$ and,*
   - *(b) $\phi : Z(R) \to M$ is the unique holomorphic/smooth map such that the following diagram commutes*

$$\begin{tikzcd} \Gamma_L \arrow[r, "p"] \arrow[dr, "t"'] & Z(R) \arrow[d, "\phi"] \\ & M \end{tikzcd} \tag{2}$$

   where $p : \Gamma_L \to Z(R) = R\backslash\Gamma_L$ is the natural projection.

2. *When $L \cap \mathcal{S}$ is a R-connected set, this étale resolution is a covering resolution with typical fiber $\frac{\pi_0(I_x(\Gamma))}{\pi_0(I_x(R))}$, where $x \in \mathcal{S}$ is an arbitrary point, and $I_x(\Gamma)$ (resp. $I_x(R)$) stands for the isotropy group of $\Gamma \rightrightarrows M$ (resp. of $R \rightrightarrows L$) at the point x.*

3. *This étale resolution is a resolution if and only if R contains $\Gamma_{L\cap\mathcal{S}}^{L\cap\mathcal{S}}$. In this case, we have $R = \overline{\Gamma_{L\cap\mathcal{S}}^{L\cap\mathcal{S}}} \cap \Gamma_L^L$.*

4. *When $L \cap \mathcal{S}$ is a connected set and $R \rightrightarrows L$ is a source-connected sub-Lie groupoid of $\Gamma \rightrightarrows M$, then the typical fiber is $\frac{\pi_1(\mathcal{S})}{j(\pi_1(L\cap\mathcal{S}))}$, where j is the map induced at the fundamental group level by the inclusion of $L \cap \mathcal{S}$ into $\mathcal{S}$.*

Example 2.12. For the algebroid crossing given in Example 2.9, the sub-Lie groupoid $R = H \times L \rightrightarrows L$ satisfies the conditions of Proposition 2.11(3). Let us describe the obtained resolution. The manifold $Z(R)$ is equal to the quotient of $\mathrm{SO}_n(\mathbb{C}) \times L$, with respect to the left H-action of H given by $h \cdot (O, l) = (hO, h(l))$ for all $h \in h, O \in \mathrm{SO}_n(\mathbb{C}), l \in L$. The map $\phi : Z(R) \to \overline{\mathcal{S}}$ is given by $\phi([g, l]) = g^{-1}(l)$, where $[g, l]$ stands for the class in $Z(R)$ of an element $(g, l) \in \mathrm{SO}_n(\mathbb{C}) \times L$.

Remark 2.13. If L itself is a closed subset of M, then R is closed as a subset of Γ_L^L if and only if it is a closed subset of Γ.

Proof. 1) Since L is a submanifold of M, and the source map $s : \Gamma \to M$ is a surjective submersion, $\Gamma_L = s^{-1}(L)$ is a submanifold of Γ. The Lie groupoid $R \rightrightarrows L$ acts on Γ_L by left multiplication. This action is free and the action map , i.e. the map,

$$\begin{array}{rcl} \mu : R \times_{t,L,s} \Gamma_L & \to & \Gamma_L \times \Gamma_L \\ \mu(r,\gamma) & = & (r \cdot \gamma, \gamma) \end{array}$$

is proper. We need to prove this last claim precisely. Let K be a compact subset of $\Gamma_L \times \Gamma_L$, and $(r_n, \gamma_n)_{n\in\mathbb{N}}$ a sequence in $R \times_{t,M,s} \Gamma_L$ with $\mu(r_n, \gamma_n) = (r_n \cdot \gamma_n, \gamma_n) \in K$. By compactness of K, one can extract a subsequence $(r_{\sigma(n)} \cdot \gamma_{\sigma(n)}, \gamma_{\sigma(n)})$ that converges to $(g_1, g_2) \in K$. Then g_1 and g_2^{-1} are composable and $(r_{\sigma(n)})_{n\in\mathbb{N}}$ converges to $r = g_1 \cdot g_2^{-1}$. Since R is closed as a subset of Γ_L^L, r is an element in R. By construction (r, g_2) is an element in $\mu^{-1}(K)$ which is therefore a compact subset of $R \times_{t,M,s} \Gamma_L$. This justifies the claim.

The action of the Lie groupoid $R \rightrightarrows L$ on Γ_L being a free and proper action, the quotient space $Z(R) := R\backslash\Gamma_L$ is a Hausdorff manifold and the projection $p : \Gamma_L \to Z(R)$ is a surjective submersion.

Since $t(r \cdot \gamma) = t(\gamma)$ for any $(r, \gamma) \in R \times_{t,L,s} \Gamma_L$, there exists an unique map $\phi : Z(R) \to M$ such that the diagram (2) commutes. Moreover, the identity $\phi(Z(R)) = t(\Gamma_L)$ holds by construction of ϕ. But, by definition of an algebroid crossing, L has a non-empty intersection with all the symplectic leaves contained in $\bar{\mathcal{S}}$, so that, by Lemma 1.3, any element of $\bar{\mathcal{S}}$ is the target of at least one element in Γ_L, and we obtain the identity $\phi(Z(L)) = t(\Gamma_L) = \bar{\mathcal{S}}$.

The identity $t(\Gamma_{L\cap\mathcal{S}}) = \mathcal{S}$ holds by Lemma 1.3. By construction of ϕ, the identity $p(\Gamma_{L\cap\mathcal{S}}) = \phi^{-1}(\mathcal{S})$ holds as well and $\phi : p(\Gamma_{L\cap\mathcal{S}}) \to \mathcal{S}$ is a surjective submersion because $t : \Gamma_{L\cap\mathcal{S}} \to \mathcal{S}$ itself a surjective submersion by Lemma 1.3 again. Now, the dimension of $Z(R)$ is given by

$$dim(Z(R)) = dim(\Gamma_L) - rk(B) = dim(L) + rk(A) - rk(B)$$

But since $\rho(B_x) = T_xL$ for all $x \in L \cap \mathcal{S}$ while $\rho(A_x) = T_x\mathcal{S}$, the ranks of the algebroids A and B are given by $rk(A) = dim(\mathcal{S}) + dim(ker(\rho))$ and $rk(B) = dim(L) + dim(ker(\rho))$ Hence $rk(A) - rk(B) = dim(\mathcal{S}) - dim(L)$ and we obtain

$$dim(Z(R)) = dim(L) - dim(L) + dim(\mathcal{S}) = dim(\mathcal{S}).$$

The dimensions of the manifolds $Z(R)$ and $\mathcal{S}$ being equal and ϕ being a surjective submersion from the open subset $\phi^{-1}(\mathcal{S}) \subset Z(R)$ onto $\mathcal{S}$, the map $\phi : \phi^{-1}(\mathcal{S}) \to \mathcal{S}$ is an étale map.

It remains to prove that $\phi^{-1}(\mathcal{S})$ is open and dense in $Z(R)$. But $L \cap \mathcal{S}$ is open in L by Lemma 2.5 and dense in L by assumption. Hence $\Gamma_{L\cap\mathcal{S}}$ is open and dense in Γ_L. Since p is a surjective submersion (and in particular an open map), $\phi^{-1}(\mathcal{S}) = p(\Gamma_{L\cap\mathcal{S}})$ is open and dense in $Z(R)$. This achieves the proof of 1).

2) By Lemma 1.3, the Lie groupoid $\Gamma \rightrightarrows M$ acts transitively on $\mathcal{S}$, hence for any $x \in \mathcal{S}$ the target map induces a biholomorphism/diffeomorphism

$$I_x(\Gamma)\backslash\Gamma_x \simeq \mathcal{S},$$

where $I_x(\Gamma)$ stands for the isotropy group at $x \in \mathcal{S}$ of the Lie groupoid $\Gamma \rightrightarrows M$.

For any $x \in L \cap \mathcal{S}$, the anchor map of the algebroid $B \to L$, is onto so that connected components of $L \cap \mathcal{S}$ are algebroid leaves for the algebroid $B \to L$, and, by Lemma 1.3 again, the action of $R \rightrightarrows L$ on $L \cap \mathcal{S}$ acts transitively on each connected component. Since $L \cap \mathcal{S}$ is R-connected, for any two connected components, there exists $r \in R$ having its source in the first one and its target in the second one. Hence the action of $R \rightrightarrows L$ on $L \cap \mathcal{S}$ is transitive, we obtain a biholomorphism/diffeomorphism

$$\phi^{-1}(\mathcal{S}) \simeq R\backslash\Gamma_{L\cap\mathcal{S}} \simeq I_x\big(R\big)\backslash\Gamma_x$$

where $I_x\big(R\big)$ stands for the isotropy group at the point $x \in L \cap \mathcal{S}$ of the Lie groupoid $R \rightrightarrows L$. Since Γ_x is a connected set, so is $\phi^{-1}(\mathcal{S}) = p(\Gamma_x)$, and the étale resolution (Z, ϕ) is a covering resolution. Moreover, the typical fiber of $\phi : \phi^{-1}(\mathcal{S}) \to \mathcal{S}$ can be identified with the quotient

$$\frac{I_x\big(\Gamma\big)}{I_x\big(R\big)}.$$

We study this quotient in detail. For any $x \in L \cap \mathcal{S}$, the kernel of the anchor maps of the algebroids $B \to L$ and $A \to M$ coincide, therefore the isotropy Lie algebras of both Lie groupoids at an arbitrary point $x \in L \cap \mathcal{S}$ coincide; therefore their isotropy Lie groups at $x \in L \cap \mathcal{S}$ have the same connected component of the identity. Denoting, for an arbitrary Lie group H, by $\pi_0(H)$ the discrete group of connected components of H, there is an isomorphism

$$\frac{I_x\big(\Gamma\big)}{I_x\big(R\big)} \simeq \frac{\pi_0(I_x\big(\Gamma\big))}{\pi_0(I_x\big(R\big))}.$$

This achieves the proof of 2)

3) The restriction to $\phi^{-1}(\mathcal{S}) \subset Z(R)$ of ϕ is an biholomorphism/diffeomorphism if and only if for any two γ_1, γ_2 in $\Gamma_{L\cap\mathcal{S}}$ with $t(\gamma_1) = t(\gamma_2)$, $\gamma_1\gamma_2^{-1}$ is an element of R. In other words, the restriction to $\phi^{-1}(\mathcal{S}) \subset Z(R)$ of ϕ is a biholomorphism/diffeomorphism if and only if $\Gamma_{L\cap\mathcal{S}}^{L\cap\mathcal{S}} \subset R$. Since R is a closed subset of Γ_L^L by assumption, this last requirement is equivalent to the requirement $\overline{\Gamma_{L\cap\mathcal{S}}^{L\cap\mathcal{S}}} \cap \Gamma_L^L \subset R$.

But, $L \cap \mathcal{S}$ being by assumption open and dense in L, $R \cap \Gamma_{L\cap\mathcal{S}}^{L\cap\mathcal{S}}$ is dense in R. In other words, $R \subset \overline{\Gamma_{L\cap\mathcal{S}}^{L\cap\mathcal{S}}}$. Since the inclusion $R \subset \Gamma_L^L$ holds also, the inclusion $R \subset \overline{\Gamma_{L\cap\mathcal{S}}^{L\cap\mathcal{S}}} \cap \Gamma_L^L$ holds.

In conclusion the inclusion $\overline{\Gamma_{L\cap\mathcal{S}}^{L\cap\mathcal{S}}} \cap \Gamma_L^L \subset R$ holds if and only if the equality $\overline{\Gamma_{L\cap\mathcal{S}}^{L\cap\mathcal{S}}} \cap \Gamma_L^L = R$ holds. This achieves the proof of 3).

4) Fix $x \in L \cap \mathcal{S}$. We have a commutative diagram of fiber bundles:

$$\begin{tikzcd}
I_x(R) \arrow[r] \arrow[d] & R_x \arrow[r, "t"] \arrow[d] & L \cap \mathcal{S} \arrow[d] \\
I_x(\Gamma) \arrow[r] & \Gamma_x \arrow[r, "t"] & \mathcal{S}
\end{tikzcd}$$

where vertical maps are inclusion maps. The usual long exact sequence in homotopy gives the following commutative diagram, where horizontal lines are exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_1(R_x) \arrow[r] \arrow[d] & \pi_1(L \cap \mathcal{S}) \arrow[r] \arrow[d, "j"] & \pi_0(I_x(R)) \arrow[r] \arrow[d] & \pi_0(R_x) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \pi_1(\Gamma_x) \arrow[r] & \pi_1(\mathcal{S}) \arrow[r] & \pi_0(I_x(\Gamma)) \arrow[r] & \pi_0(\Gamma_x) \arrow[r] & \cdots
\end{tikzcd}$$

By assumption, Γ_x and R_x are connected while Γ_x is simply-connected. Hence, in the central square of the previous diagram, that is to say, in the diagram

$$\begin{tikzcd}
\pi_1(L \cap \mathcal{S}) \arrow[r] \arrow[d, "j"] & \pi_0(I_x(R)) \arrow[d] \\
\pi_1(\mathcal{S}) \arrow[r] & \pi_0(I_x(\Gamma))
\end{tikzcd}$$

the upper line is a surjective group morphism, the lower line is a group isomorphism, while the right vertical line is an injective group morphism (since it is induced by the inclusion). The proof of 4) now follows by an easy diagram chasing. $\square$

At first glance, it seems that resolutions obtained from the previous construction depends strongly on the choice of an algebroid crossing. We study this dependence using the notion of Morita equivalence of Lie groupoids. This notion of Morita equivalence goes back to [13], but we invite the reader to consult [18] for an introduction to the notion of Morita equivalence of modules of Lie groupoids that matches the presentation below.

Proposition 2.14. *Let $(A \to M, \rho, [\cdot,\cdot])$ be a Lie algebroid, $\mathcal{S}$ a locally closed orbit of this algebroid, and $L_i, i = 1,2$ two algebroid crossings of $\bar{\mathcal{S}}$ with normalisations $B_i \to L_i$. Assume that:*

1. *there exists a source-connected Hausdorff Lie groupoid $\Gamma \rightrightarrows M$ integrating the Lie algebroid $(A \to M, \rho, [\cdot,\cdot])$ and*
2. *there exists, for $i = 1,2$, sub-Lie groupoids $R_i \rightrightarrows L_i$ of $\Gamma \rightrightarrows M$, closed as a subset of $\Gamma_{L_i}^{L_i}$, integrating the Lie algebroid $(B_i \to L_i, \rho, [\cdot,\cdot])$, and containing $\Gamma_{L_i\cap\mathcal{S}}^{L_i\cap\mathcal{S}}$.*

Let (Z_i, ϕ_i), $i = 1,2$ be the two corresponding resolutions as in Proposition 2.11(3).

The following are then equivalent:

(i) the resolutions (Z_1, ϕ_1) and (Z_2, ϕ_2) are isomorphic,

(ii) $\overline{\Gamma^{L_2\cap\mathcal{S}}_{L_1\cap\mathcal{S}}} \cap \Gamma^{L_2}_{L_1}$ *is a submanifold of* Γ*, and the restrictions to this submanifold of the source and the target maps are surjective submersions onto* L_1 *and* L_2 *respectively,*

(iii) there exists a submanifold I *of* Γ *that gives a Morita equivalence between the Lie groupoids* $R_1 \rightrightarrows L_1$ *and* $R_2 \rightrightarrows L_2$*:*

$$\begin{tikzcd}
\Gamma_{L_1} \arrow[dr] & R_1 \arrow[d, shift left] \arrow[d, shift right] & I \arrow[dl, "s"] \arrow[dr, "t"'] & R_2 \arrow[d, shift left] \arrow[d, shift right] & \Gamma_{L_2} \arrow[dl] \\
& L_1 & & L_2 &
\end{tikzcd}$$

In this case moreover, the R_1*-module* Γ_{L_1} *corresponds to the* R_2*-module* Γ_{L_2} *with respect to the Morita equivalence* I*.*

We need some preliminary results. Let $(Z(R) = R\backslash\Gamma_L, \phi)$ be an étale resolution as in Proposition 2.11(1). There is a natural map j from L to $Z(R)$ obtained by composing the restriction to L of the unit map $\varepsilon : L \to \Gamma_L$ with the canonical projection $p : \Gamma_L \to Z(R)$. By Equation (2), ϕ is a left inverse of j, i.e. $\phi \circ j = \mathrm{id}_L$, so that $j(L)$ is a submanifold of $Z(R)$. In short:

Lemma 2.15. *Let* $(Z(R) = R\backslash\Gamma_L, \phi)$ *be an étale resolution as in Proposition 2.11 and* $j : L \to Z(R)$ *as above. Then* $j(L)$ *is a submanifold of* $Z(R)$ *and the restriction of* ϕ *to* $j(L)$ *is a biholomorphism/diffeomorphism onto* L*.*

We now prove Proposition 2.14.

Proof. We say simply diffeomorphism rather than biholomorphism/diffeomorphism.

(i) $\Longrightarrow$ *(ii).* Denote by $p_i, i = 1, 2$ and $j_i, i = 1, 2$ the projections from $\Gamma_{L_i} \to Z_i$ and the inclusions of L_i in Z_i respectively.

Let $\Psi : Z_1 \to Z_2$ be a diffeomorphism of resolutions from (Z_1, ϕ_1) to (Z_2, ϕ_2). Then $\Psi \circ p_1 : \Gamma_{L_1} \to Z_2$ is a surjective submersion. The inverse image I_1 of the submanifold $j_2(L_2)$ of Z_2 by $\Psi \circ p_1$ is a submanifold of Γ_{L_1}, which is closed as a subset of $\Gamma^{L_2}_{L_1}$, and, by Lemma 2.15, $t = \phi_2 \circ \Psi \circ p_1 : I_1 \to L_2$ is a surjective submersion. In particular, I_1 contains $\Gamma^{L_2\cap\mathcal{S}}_{L_1\cap\mathcal{S}}$ as a dense subset. Hence $I_1 = \overline{\Gamma^{L_2\cap\mathcal{S}}_{L_1\cap\mathcal{S}}} \cap \Gamma^{L_2}_{L_1}$ and the latter is a submanifold of Γ.

Similarly, since $\Psi^{-1} : Z_2 \to Z_1$ is a diffeomorphism, $\Psi^{-1} \circ p_2 : \Gamma_{L_2} \to Z_1$ is a surjective submersion. The inverse image I_2 of the submanifold $j_1(L_1)$ of Z_1 by $\Psi^{-1} \circ p_2$ is equal to $\overline{\Gamma^{L_1\cap\mathcal{S}}_{L_2\cap\mathcal{S}}} \cap \Gamma^{L_1}_{L_2}$ and $t : I_2 \to L_1$ is surjective submersion.

In particular, we obtain $I_2 = I_1^{-1}$ and, since the inverse map intertwines sources and targets, $s : I_1 \to L_1$ is also a surjective submersion. This completes the proof of *(i)* $\Longrightarrow$ *(ii).*

(ii) $\Longrightarrow$ *(iii).* Let $I = \overline{\Gamma^{L_2\cap\mathcal{S}}_{L_1\cap\mathcal{S}}} \cap \Gamma^{L_2}_{L_1}$. For any $m \in L_1 \cap \mathcal{S}$, the Lie groupoid $R_2 \rightrightarrows L_2$ acts transitively on the fiber of $s : I \to L_1$ over m, since this fiber is precisely equal to $\Gamma^{L_2\cap\mathcal{S}}_m$ while R_2 contains $\Gamma^{L_2\cap\mathcal{S}}_{L_2\cap\mathcal{S}}$. Let us show that this fact remains true for all $m \in L_1$. Let $c, c' \in I$ be two points in I with $s(c) = m = s(c')$. There exist sequences $(c_n)_{n\in\mathbb{N}}, (c'_n)_{n\in\mathbb{N}}$ of elements of I, converging to c and c' respectively, and which satisfy $s(c_n) = s(c'_n)$ for all $n \in \mathbb{N}$. The sequence $(c'_n)^{-1}c_n$ is a sequence of R_2 that converges to $(c')^{-1}c$. Since

R_2 is closed in $\Gamma_{L_2}^{L_2}$, we have $(c')^{-1}c \in R_2$, and $R_2 \rightrightarrows L_2$ acts transitively on the fibers of $s : I \to L_1$. This action is also, of course, free.

Similarly, the fibers of $t : I \to L_1$ are precisely the fibers of the R_1-action. The left and right actions of $R_1 \rightrightarrows L_1$ and $R_2 \rightrightarrows L_2$ is free and proper, with $I_1 \simeq I/R_2$ and $R_1\backslash I \simeq l_2$. Hence I is a Morita bimodule that gives a Morita equivalence between the Lie groupoids $R_1 \rightrightarrows L_1$ and $R_2 \rightrightarrows L_2$.

We now prove that, in this case moreover, the R_1-module Γ_{L_1} corresponds to the R_2-module Γ_{L_2} with respect to the Morita equivalence I. W e recall some general facts about Morita equivalences (see [18] or [26]). A Morita bimodule between two Lie groupoids induces a one-to-one correspondence between their left modules that we now describe. Let (X_2, χ_2) be a left R_2-module. Then the diagonal action of $R_2 \rightrightarrows L_2$ on $I \times_{s,L_2,\chi_2} X$ is free and proper, so that the quotient space

$$X_1 := \frac{I \times_{s,L_2,\chi_2} X_2}{R_2} = \frac{\{(c,x) \in I \times X_2 \,|\, t(c) = \chi_2(x)\}}{(c,x) \sim (cr^{-1}, rx)}$$

is a manifold again. The map $t : I \to L_2$ factorizes to yield a surjective submersion $\chi_1 : X_1 \to L_1$. The left action of the Lie groupoid $R_1 \rightrightarrows L_1$ on I induces a left action of $R_1 \rightrightarrows L_1$ on (X_1, χ_1) which gives the desired structure of left R_1-module.

In the particular case where $(X_2, \chi_2) = (\Gamma_{L_2}, s)$ and the Morita equivalence is with respect to I, we easily check that X_1 is isomorphic to $Z_1 = R_1\backslash\Gamma_{L_1}$, the isomorphism in question being simply given by

$$[c, x_2] \to p_1(cx_2)$$

where (c, x_2) is an element of $I \times_{s,L_2,s} \Gamma_{L_2}$, $[c, x_2]$ its class in $\frac{I \times_{s,L_2,\chi_2} X_2}{R_2}$ and $p_1 : \Gamma_{L_1} \to Z_1 = R_1\backslash\Gamma_{L_1}$ the canonical projection. This proves 3).

(iii) $\Longrightarrow$ *(i)*. Under the correspondence between left R_1- and R_2-modules just described above, when $X_2 = R_2\backslash\Gamma_{L_2}$ is a manifold, then so is $X_1 = R_1\backslash\Gamma_{L_1}$ and both manifolds are canonically isomorphic (see [26]). The isomorphism Ψ in question is given by $[x_2] \to [(i, x_2)]$, where $x_2 \in X_2$ is an arbitrary element, $[x_2]$ its class in the quotient space $R_2\backslash X_2$, $i \in I$ is any element with $t(i) = \phi_2(x_2)$, and $[i, x_2]$ is the class of (i, x_2) in the quotient space $R_1\backslash X_1$. In the particular case $X_2 = \Gamma_{L_2}$ and $\chi_2 = s$, Ψ maps $R_2\backslash\Gamma_{L_2}$ to $R_1\backslash\Gamma_{L_1}$. Moreover, Ψ satisfies by construction the relation $\phi_1 \circ \Psi = \phi_2$ and is therefore an isomorphism of resolutions. $\square$

3 Resolution compatible with a multi-vector field.

3.1 Definition of a resolution compatible with a multi-vector field

By an abuse of language (justified by its use in algebraic geometry), a k-vector field π_M on the manifold M is said to be *tangent to* $\bar{\mathcal{S}}$ if it is tangent to $\mathcal{S}$, i.e. if $\pi_M[f, f_2, \cdots, f_k]$ vanishes on $\mathcal{S}$ for any local function f that vanishes on $\mathcal{S}$ (equivalently on $\overline{\mathcal{S}}$), and any local functions $f_2, \cdots, f_k \in \mathcal{F}(M)$.

Let us now explain what we mean by a resolution compatible with a k-vector field π_M tangent to $\bar{\mathcal{S}}$.

Definition 3.1. *Let $\bar{\mathcal{S}}$ be the closure of a complex/real locally closed (= embedded) submanifold $\mathcal{S}$ of M and π_M a k-vector field tangent to $\bar{\mathcal{S}}$, a* resolution compatible with the k-vector field π_M *is a pair (Z, ϕ) where*

1. *(Z, ϕ) is a resolution of $\bar{\mathcal{S}}$,*
2. *the k-vector field π_Z defined on $\phi^{-1}(\mathcal{S})$ by $\phi_* \pi_Z = \pi_M$ extends to a holomorphic/smooth k-vector field on Z.*

When (Z, ϕ) is only an étale/covering resolution, then we speak of an étale/covering resolution compatible with the k-vector field π_M. *If moreover, π_M is a Poisson bivector field, we speak of a* Poisson resolution.

We denote by π_Z again the extension to Z of π_Z. For convenience, we often denote a compatible resolution by a triple (Z, ϕ, π_Z). But we invite the reader to keep in mind that, for a compatible étale resolution, π_Z is indeed determined by (Z, ϕ).

Remark 3.2. For any resolution compatible with π_M, the relation $\phi_* \pi_Z = \pi_M$ holds indeed on Z by density of $\phi^{-1}(\mathcal{S})$ in Z. In particular, when π_M is a Poisson bivector field, so is π_Z.

For clarity, we list all the conditions required in order to have an étale resolution (Z, π_Z, ϕ) of the closure $\bar{\mathcal{S}}$ of a locally closed submanifold of M which is compatible with a k-vector field π_M tangent to $\bar{\mathcal{S}}$.

1. Z is a manifold and π_Z a k-vector field,
2. $\phi : Z \to M$ is a holomorphic/smooth map from the manifold Z to the manifold M,
3. $\phi(Z) = \bar{\mathcal{S}}$,
4. $\phi^{-1}(\mathcal{S})$ is open and dense in Z (note that density is automatically satisfied in the complex case),
5. the restriction of ϕ to a map from $\phi^{-1}(\mathcal{S})$ to $\mathcal{S}$ is a biholomorphism/diffeomorphism,
6. $\phi : Z \to M$ maps π_Z to π_M.

For étale/covering symplectic resolutions, the fourth point needs to be replaced by "the restriction of ϕ to $\phi^{-1}(\mathcal{S})$ is an étale/covering map over $\mathcal{S}$".

The following Lemma will be useful in the next section. Notations are as before.

Lemma 3.3. *Let (Z_i, ϕ_{Z_i}), with $i = 1, 2$, be étale resolutions of $\bar{\mathcal{S}}$, and $\Psi : Z_1 \to Z_2$ a local diffeomorphism which is a morphism of resolutions of $\bar{\mathcal{S}}$*

$$\begin{tikzcd} Z_1 \arrow[r, "\Psi"] \arrow[dr, "\phi_{Z_1}"'] & Z_2 \arrow[d, "\phi_{Z_2}"] \\ & \bar{\mathcal{S}} \end{tikzcd}$$

Then, if (Z_1, ϕ_{Z_1}) is an étale resolution compatible with π_M, then (Z_2, ϕ_{Z_2}) is also an étale resolution compatible with π_M.

Proof. We denote by π_{Z_1} the unique k-vector field on Z_1 satisfying $(\phi_{Z_1})_* \pi_{Z_1} = \pi_M$. For any two points $x, y \in Z_1$ with $\Psi(x) = \Psi(y)$, there exists a local biholomorphism/diffeomorphism Φ of Z_1 over the identity of Z_2, (id est $\Psi \circ \Phi = \Psi$) defined on some open subset $U \subset Z_1$ with $\Phi(x) = y$. The relation $\Psi_*(\pi_{Z_1})_{|_x} = (\pi_{Z_1})_{|_y}$ holds for any $x \in U \cap \phi_1^{-1}(\mathcal{S})$ since

$$(\phi_{Z_1})_*(\pi_{Z_1})_{|_x} = (\pi_M)_{|_{\phi_{Z_1}(x)}} = (\phi_{Z_1})_*(\pi_{Z_1})_{|_y}$$

and since $(\phi_{Z_1})_* : T_m Z_1 : T_{\phi_{Z_1}(m)}\mathcal{S}$ is invertible for all $x \in U \cap \phi_1^{-1}(\mathcal{S})$. The open subset $U \cap \phi_{Z_1}^{-1}(\mathcal{S})$ being dense in U, the relation $\Phi_*(\pi_{Z_1})_{|_x} = (\pi_{Z_1})_{|_y}$ holds for any $x \in U$. This amounts to the fact that π_{Z_1} goes to the quotient and defines a k-vector field π_{Z_2} on Z_2. By construction, $\Psi_* \pi_{Z_1} = \pi_{Z_2}$. It is immediate that

$$(\phi_{Z_2})_* \, \pi_{Z_2} = (\phi_{Z_2})_* \circ \Psi_* \, \pi_{Z_1} = (\phi_{Z_1})_* \, \pi_{Z_1} = \pi_M.$$

The k-vector field π_{Z_2} is defined everywhere on Z_2 and projects on π_M through $(\phi_2)_*$, so that the étale resolution (Z_2, ϕ_2) is compatible with π_M. $\square$

3.2 Lie groupoids, multiplicative multivector fields and compatible resolutions of the closure of an algebroid leaf

Recall (see Definition 2.6 in [15]) that a k-vector field on a Lie groupoid $\Gamma \rightrightarrows M$ is said to be multiplicative when the graph of the multiplication of the groupoid, that is to say the submanifold of Γ^3 given by

$$\mathrm{Gr}(\Gamma) = \{(\gamma_1, \gamma_2, \gamma_1\gamma_2) \in \Gamma^3 | t(\gamma_1) = s(\gamma_2)\},$$

is coisotropic with respect to $\pi_\Gamma \oplus \pi_\Gamma \oplus (-1)^{k+1}\pi_\Gamma$.

Multiplicative vector fields on Lie groupoids have a very rich geometry. We invite the reader to read Section 2 in [15] to get an overview of that difficult matter, and recall two points of fundamental importance for the present purpose.

We recall (see Remark 2.4 in [15]) that a submanifold N of M is *coisotropic with respect to a k-vector field* $\pi_M \in \mathcal{X}^k(M)$ if and only if, for any local functions $f_1, \dots, f_k \in \mathcal{F}(M)$ that vanish on N, the function $\pi_M[f_1, \dots, f_k]$ vanishes on N.

The purpose of the present section is to

(i) recall from [15] how this multiplicative k-vector field π_Γ induces a k-vector field π_M on M, and explain why, for $k \geq 2$, this k-vector field is tangent to $\bar{\mathcal{S}}$,

(ii) show that any resolution $(Z(R) = R \backslash \Gamma_L, \phi)$ of an algebroid leaf $\overline{\mathcal{S}}$ constructed with the help of an algebroid crossing L as in Proposition 2.11 is compatible with π_M under the condition that L is coisotropic with respect to π_M.

(i) **From multiplicative k-vector field to k-vector field on $\mathcal{S}$.**

We prove in this section that a multiplicative k-vector field on $\Gamma \rightrightarrows M$ induces a k-vector field on M which is tangent to all algebroid leaves for $k \geq 2$, see Proposition 3.4 below. We characterise, in term of k-differentials on the Lie algebroid, algebroid crossing which are coisotropic with respect to this induced k-vector field, see Proposition 3.5 below.

To start with, according to Proposition 2.21 in [15], there exists a unique k-vector field π_M on M such that

$$s_*\pi_\Gamma = \pi_M \text{ and } t_*\pi_\Gamma = (-1)^{k+1}\pi_M. \tag{3}$$

We now prove the following Proposition.

Proposition 3.4. *For any multiplicative k-vector field π_Γ on $\Gamma \rightrightarrows M$ with $k \geq 2$, the k-vector field $\pi_M = s_*\pi_\Gamma$ is tangent to all the algebroid leaves.*

Proof. According to Propositions 2.17,2.18 and Corollary 2.20 in [15], there exists a k-differential $\delta : \Gamma(\wedge^\bullet A \to M) \to \Gamma(\wedge^{\bullet+k-1}A \to M)$ such that,

$$\begin{cases} [\![\pi_\Gamma, \overrightarrow{a}]\!]_{T\Gamma} & = & \overrightarrow{\delta(a)} \\ [\![\pi_\Gamma, \overleftarrow{a}]\!]_{T\Gamma} & = & \overleftarrow{\delta(a)} \end{cases} \tag{4}$$

where $\overrightarrow{a}$ (resp. $\overleftarrow{a}$) is the right-invariant (resp. left-invariant) multi-vector field on Γ corresponding to $a \in \Gamma(\wedge^l A \to M)$, with the understanding that $\overrightarrow{f} = s^*f$ (resp. $\overleftarrow{f} = t^*f$) when f is a function on M (see Section 1.3 and especially Equation (1) for the definition of a k-differential).

Note that [15] deals with real Lie groupoids only, but these results extend to the complex setting without any difficulty, with the understanding, however, that $\Gamma(\wedge^\bullet A \to M)$ stands for the sheaf of local sections.

For any local functions $g_1, \cdots, g_{k-1}, f$ on M, with f vanishing on $\mathcal{S}$, we have, in view of Equation (3)

$$s^*(\pi_M[g_1, \cdots, g_{k-1}, f]) = \pi_\Gamma[s^*g_1, \ldots, s^*g_{k-1}, s^*f] = [\![\pi_\Gamma, s^*g_1]\!]_{T\Gamma}[s^*g_2, \cdots, s^*g_{k-1}, s^*f].$$

Equation (4) yields

$$s^*(\pi_M[g_1, \cdots, g_{k-1}, f]) = \overrightarrow{\delta(g_1)}[s^*g_2, \cdots, s^*g_{k-1}, s^*f].$$

But, in turn, we have

$$\overrightarrow{\delta(g_1)}[s^*g_2, \cdots, s^*g_{k-1}, s^*f] = s^*\left(\rho(\delta(g_1))[g_2, \cdots, g_{k-1}, f]\right).$$

Since f vanishes on $\mathcal{S}$, and since $\rho(\delta(f_1))$ is tangent to all the algebroid leaves, $\rho(\delta(g_1))[g_2, \cdots, g_{k-1}, f]$ vanishes on $\mathcal{S}$ as well. In conclusion, the function

$$\pi_M[g_1, \cdots, g_{k-1}, f]$$

vanishes on $\mathcal{S}$ for any local functions $g_1, \cdots, g_{k-1}$ provided that f vanishes on $\mathcal{S}$. Hence π_M is tangent to $\mathcal{S}$. $\square$

Assume now that we are given an algebroid crossing L of $\overline{\mathcal{S}}$ with normalisation $B \to L$, where $\mathcal{S}$ is a locally closed (= embedded) algebroid leaf. We now explore the picture for L coisotropic with respect to π_M.

We say that a k-differential $\delta : \Gamma(\wedge^\bullet A_{|_U} \to U) \to \Gamma(\wedge^{\bullet+k-1}A_{|_U} \to U)$ is *compatible with an algebroid crossing L with normalization $B \to L$* if and only if $\delta(\langle B\rangle_U) \subset \langle B\rangle_U$ for all

open subset $U \subset M$, where $\langle B\rangle_U$ is the ideal in $(\Gamma(\wedge^\bullet A_{|U} \to U), \wedge)$ generated by functions vanishing on L and sections of A whose restriction to L is a section of $B \to L$.

In other words, a local section X of $\Gamma(\wedge^p A_{|U} \to U)$ belongs to $\langle B\rangle_U$ if, and only if, for all $m \in L \cap U$, $X_{|m}$ is a linear combination of terms of the form $b \wedge a_1 \wedge \cdots \wedge a_{p-1}$, with $b \in B_m$, $a_1, \cdots, a_{p-1} \in A_m$.

The need of the following Proposition will appear in the sequel.

Proposition 3.5. *Let $k \geq 2$ be an integer and $\Gamma \rightrightarrows M$ a Lie groupoid endowed with a multiplicative k-vector field π_Γ. Denote by π_M the k-vector field on M induced by π_Γ by Equation (3), and, by δ the k-differential δ induced by Equation (4).*

Let $\mathcal{S}$ be a locally closed algebroid leaf. An algebroid crossing L of $\overline{\mathcal{S}}$, with normalisation $B \to L$, is compatible with the k-differential δ if and only if L is coisotropic with respect to π_M.

The following Lemma is an immediate consequence of the density of $L \cap \mathcal{S}$ in L and from the defining relation $B_m = \rho^{-1}(T_m L)$ for all $m \in L \cap \mathcal{S}$.

Lemma 3.6. *Let $U \subset M$ be an open subset.*

1. *For any homogeneous element $X \in \langle B\rangle_U$ of degree ≥ 1, $L \cap U$ is coisotropic with respect to $\rho(X)$.*
2. *Conversely, any section $X \in \Gamma(\wedge^i A_{|U} \to U)$ with $i \geq 1$ such that $L \cap U$ is coisotropic with respect to $\rho(X)$ is a section of $\langle B\rangle_U$.*

Now, we can turn our attention to the proof of Proposition 3.5.

Proof. Let $U \subset M$ be a trivial open subset of M. Assume that $B \to L$ is compatible with the k-differential δ. To start with, we recall from Lemma 2.32 in [15] that the following identity holds

$$\pi_M[f_1, \ldots, f_k] = (-1)^{k+1}\rho(\delta(f_1))[f_2, \ldots, f_k] \tag{5}$$

for any local functions $f_1, \cdots, f_k \in \mathcal{F}(U)$. Now, let $f_1, \ldots, f_k$ be functions that vanish on $L \cap U$. The k-differential δ being compatible with the algebroid crossing $B \to L$, we have $\delta(f_1) \in \langle B\rangle_U$. By Lemma 3.6-(1), $L \cap U$ is coisotropic with respect to $\rho(\delta(f_1))$, so that the function $\pi_M[f_1, \ldots, f_k] = (-1)^{k+1}\rho(\delta(f_1))[f_2, \ldots, f_k]$ vanishes on $L \cap U$. In conclusion, L is coisotropic with respect to π_M.

Conversely, assume that L is coisotropic with respect to π_M. Let $f \in \mathcal{F}(U)$ be a local function that vanishes on $L \cap U$. It is immediate that L is also coisotropic with respect to $[\![\pi_M, f]\!]_{TM}$. By Eq. (5), the identity $[\![\pi_M, f]\!]_{TM} = (-1)^{k+1}\rho\big(\delta(f)\big)$ holds, so that, according to Lemma 3.6 (2), we have $\delta(f) \in \langle B\rangle_U$.

At this point, to show that $\delta(\langle B\rangle_U) \subset \langle B\rangle_U$, it suffices to show that $\delta(b) \in \langle B\rangle_U$ for any section $b \in \langle B\rangle_U$. According to Lemma 3.6 (2), it suffices indeed to prove that $\rho(\delta(b))[f_1, \cdots, f_k]$ vanishes on $L \cap U$ if $f_1, \cdots, f_k \in \mathcal{F}(U)$ vanish on $L \cap U$. Let us prove this point

$$\begin{aligned}
\rho\left(\delta(b)\right)[f_1, \cdots, f_k] &= [\![\rho(\delta(b)), f_1]\!]_{TM}[f_2, \cdots, f_k] \\
&= \rho\left([\![\delta(b), f_1]\!]_A\right)[f_2, \cdots, f_k] \\
&= \rho\left(\delta([\![b, f_1]\!]_A)\right)[f_2, \cdots, f_k] \\
&\quad -\rho\left([\![b, \delta(f_1)]\!]_A\right)[f_2, \cdots, f_k] \quad \text{by Eq. (1)}
\end{aligned}$$

Now, $[\![b, f_1]\!]_A = \rho(b)[f_1]$ is a function that vanishes on $L \cap U$. Hence $\delta([\![b, f_1]\!]_A)$ is a section in $\langle B \rangle_U$, so that L is coisotropic with respect to $\rho\left(\delta([\![b, f_1]\!]_A)\right)$ by Lemma 3.6, and the function

$$\rho\left(\delta([\![b, f_1]\!]_A)\right)[f_2, \cdots, f_k]$$

vanishes on $L \cap U$. Also, we have $\rho\left([\![b, \delta(f_1)]\!]_A\right) = [\![\rho(b), \rho(\delta(f_1))]\!]_{TM}$; which implies, since the Schouten bracket of coisotropic multi-vector fields is again coisotropic, that L is coisotropic with respect to $\rho\left([\![b, \delta(f_1)]\!]_A\right)$, and the function

$$\rho\left([\![b, \delta(f_1)]\!]_A\right)[f_2, \cdots, f_k]$$

vanishes on $L \cap U$. These two last equations imply that the function $\rho\left(\delta(b)\right)[f_1, \cdots, f_k]$ vanishes on $L \cap U$. This completes the proof. $\square$

(ii) **From a multiplicative k-vector field on $\Gamma \rightrightarrows M$ to a k-vector field on $Z(R)$.**

We prove in this section that, given a multiplicative k-vector field π_Γ on the Lie groupoid $\Gamma \rightrightarrows M$, the étale resolutions constructed in Proposition 2.11 are automatically compatible with $\pi_M = s_* \pi_\Gamma$, provided that the chosen algebroid crossing, with respect to which they are constructed, is coisotropic with respect to π_M. Roughly speaking, the idea is to construct explicitly the k-vector field on the resolution $(Z(R), \phi)$ out of π_Γ by the kind of reduction procedure described in Lemma 3.8 below.

The following Theorem summarises what this section adds to Proposition 2.11.

Theorem 3.7. *Let $(A \to M, \rho, [\cdot, \cdot])$ be a Lie algebroid, $\mathcal{S}$ a locally closed algebroid leaf and L an algebroid crossing of $\overline{\mathcal{S}}$ with normalisation $B \to L$. If*

1. *there exists a source-connected Hausdorff Lie groupoid $\Gamma \rightrightarrows M$ integrating $(A \to M, \rho, [\cdot, \cdot])$, and*
2. *there exists a multiplicative k-vector field π_Γ, with $k \geq 2$, on $\Gamma \rightrightarrows M$ such that L is coisotropic with respect to $s_* \pi_\Gamma = \pi_M$, and*
3. *there exists a sub-Lie groupoid $R \rightrightarrows L$ of $\Gamma \rightrightarrows N$, closed as a subset of Γ_L^L, integrating the algebroid $(B \to M, \rho, [\cdot, \cdot])$,*

then

1. *π_M is well-defined and tangent to $\mathcal{S}$.*
2. *$(Z(R), \phi, \pi_{Z(R)})$ is an étale resolution of $\bar{\mathcal{S}}$ compatible with π_M, where*
   - *(a) $Z(R) = R \backslash \Gamma_L$ and,*
   - *(b) $\phi : Z(R) \to M$ is the unique holomorphic/smooth map such that the following diagram commutes*

$$\begin{tikzcd} \Gamma_L \arrow[r, "p"] \arrow[dr, "t"'] & Z(R) \arrow[d, "\phi"] \\ & M \end{tikzcd} \tag{6}$$

   where $p : \Gamma_L \to Z(R) = R \backslash \Gamma_L$ is the natural projection, and

(c) $\pi_{Z(R)}$ *is the unique k-vector field on M such that for all local functions $\tilde{f}_1, \cdots, \tilde{f}_k \in \mathcal{F}(\Gamma)$ and $f_1, \cdots, f_k \in \mathcal{F}(Z(R))$ the relations*

$$\begin{cases} p^* f_1 &= \imath^* \tilde{f}_1 \\ &\vdots \\ p^* f_k &= \imath^* \tilde{f}_k \end{cases}$$

imply

$$p^* \pi_{Z(R)}[f_1, \cdots, f_k] = \imath^* \pi_\Gamma[\tilde{f}_1, \cdots, \tilde{f}_n] \tag{7}$$

(wherever these identities make sense). Here $\imath$ stands for the inclusion map $\Gamma_L \hookrightarrow \Gamma$.

(d) *When $(\Gamma \rightrightarrows M, \pi_\Gamma)$ is a Poisson Lie groupoid, id est $k = 2$ and π_Γ is a multiplicative Poisson bivector field on $\Gamma \rightrightarrows M$, then $\pi_{Z(R)}$ is a Poisson bivector field and the resolution $(Z(R), \phi, \pi_{Z(R)})$ is a Poisson resolution.*

3. *When $L \cap \mathcal{S}$ is a R-connected set, this étale resolution is a covering resolution with typical fiber $\frac{\pi_0(I_x(\Gamma))}{\pi_0(I_x(R))}$, where $x \in \mathcal{S}$ is an arbitrary point, and $I_x(\Gamma)$ (resp. $I_x(R)$) stands for the isotropy group of $\Gamma \rightrightarrows M$ (resp. of $R \rightrightarrows L$) at the point x.*

4. *This étale resolution is a resolution if and only if $R = \overline{\Gamma^{L\cap\mathcal{S}}_{L\cap\mathcal{S}}} \cap \Gamma^L_L$.*

5. *When $L \cap \mathcal{S}$ is a connected set and $R \rightrightarrows L$ is a source-connected sub-Lie groupoid of $\Gamma \rightrightarrows M$, then the typical fiber is isomorphic to $\frac{\pi_1(\mathcal{S})}{j(\pi_1(L\cap\mathcal{S}))}$, where j is the map induced at the fundamental group level by the inclusion of $L \cap \mathcal{S}$ into $\mathcal{S}$.*

We postpone until the end of this section the proof of Lemma 3.8 below, which gives us the general frame to operate reduction of multi-vector fields. For a submanifold N of a manifold Q and a point $x \in N$, $T_xN^\perp \subset T^*_xQ$ stands for $\{\alpha \in T^*_xQ \,|\, T_xN \subset \ker(\alpha)\}$.

Lemma 3.8. *Let Q be a manifold, $\imath : N \hookrightarrow Q$ a submanifold of Q and $\Phi : N \to P$ a surjective submersion with connected fibers. Let π_Q be a k-vector field on M. If*

1. $(\pi_Q)_{|x} \left[\wedge^{k-1}(\ker(\mathrm{d}_x\Phi))^\perp \wedge T_xN^\perp\right] = 0$ *and,*

2. *for any $n \in N$ and $u \in T_nN$, there exists a vector field X on Q tangent to N, whose value at n is u, and such that $(L_X\pi_Q)_{|x}[\wedge^k(\ker(\mathrm{d}_x\Phi))^\perp] = 0$.*

then there exists a unique k-vector field π_P on P such that, for any local functions $\tilde{f}_1, \cdots, \tilde{f}_k$ in $\mathcal{F}(Q)$ on N and $f_1, \cdots, f_k$ in $\mathcal{F}(P)$, the relations

$$\begin{cases} \Phi^* f_1 &= \imath^* \tilde{f}_1 \\ &\vdots \\ \Phi^* f_k &= \imath^* \tilde{f}_k \end{cases}$$

imply

$$\Phi^* \pi_P(f_1, \cdots, f_k) = (-1)^{k+1} \imath^* \pi_Q[\tilde{f}_1, \cdots, \tilde{f}_n] \tag{8}$$

(wherever these identities make sense).

We now prove Theorem 3.7.

Proof. Theorem 3.7(1) is nothing but the statement of Proposition 3.4. Points (3)-(5) are a consequence of Theorem 3.7(2) and Proposition 2.11(2)-(4). It remains us, therefore, the task of proving Theorem 3.7(2). Now, when the assumptions of Theorem 3.7(2) are satisfied for a sub-Lie groupoid $R \rightrightarrows L$ of $\Gamma \rightrightarrows M$ integrating $B \to L$, all the assumptions of Proposition 2.11(1) are satisfied as well. Therefore, it only remains us the task of proving that the multivector field $\pi_{Z(R)}$ that appears in Eq. (7) can indeed be constructed.

To start with, we consider $R_0 \rightrightarrows L$ the connected component of the identities of the groupoid $R \rightrightarrows L$. Again, $R_0 \rightrightarrows L$ is a sub-Lie groupoid of $\Gamma \rightrightarrows M$ that integrates $B \to L$ and which is closed in Γ_L^L.

Let us check that the assumptions of Lemma 3.8 are satisfied with $Q := \Gamma, N := \Gamma_L, P := Z(R_0), \pi_Q := \pi_\Gamma$ and $\Phi := p$ the natural projection $\Gamma_L \to Z(R_0) = R_0 \backslash \Gamma_L$. First, since $R_0 \rightrightarrows L$ is source-connected, p is a surjective submersion with connected fibers. The two points below show that the assumptions 1 and 2 in Lemma 3.8 are satisfied.

1. The identity
$$(\pi_\Gamma)_{|_\gamma} \left[\wedge^{k-1} (\ker(\mathrm{d}_\gamma p))^\perp \wedge (T_\gamma \Gamma_L)^\perp \right] = 0 \tag{9}$$
holds for any $\gamma \in \Gamma_L$. Since $s^* : T_m L^\perp \to T_\gamma \Gamma_L^\perp$ is one-to-one, for any $\alpha \in (T_\gamma \Gamma_L)^\perp$, there exists a local function f on M, defined in a neighbourhood of $s(\gamma)$ and vanishing on L, such that $\mathrm{d}_\gamma s^* f = s^* \mathrm{d}_{s(\gamma)} f = \alpha$. Now, Eq. (4) gives
$$[\![\pi_\Gamma, s^* f]\!]_{T\Gamma} = [\![\pi_\Gamma, \overrightarrow{f}]\!]_{T\Gamma} = \overrightarrow{\delta(f)}.$$
By assumption, L is coisotropic with respect to π_M, and it follows from Proposition 3.5 that $\delta(f)$ is a section of $\langle B \rangle_U$. In particular, $\overrightarrow{\delta(f)}_\gamma$ lies in the ideal of $\wedge^\bullet T_\gamma \Gamma$ (with respect to the wedge product) generated by the space of elements of the form $\overrightarrow{b}_{|_\gamma}$ with $b \in B_{s(\gamma)}$. But this space is, by definition, the foliation given by the left action of $R_0 \rightrightarrows L$ on Γ_L, and coincides therefore with the kernel of $\mathrm{d}_\gamma p$. This amounts to the fact that $\overrightarrow{\delta(f)}_{|_\gamma}(\beta) = 0$ for any $\beta \in \wedge^{k-1}(\ker(\mathrm{d}_\gamma p))^\perp$. Hence $(\pi_\Gamma)_{|_\gamma}(\alpha \wedge \beta) = 0$. This proves Eq. (9) and Condition 1 in Lemma 3.8 is satisfied.

2. For any $\gamma \in T_\gamma \Gamma_L$ and any $u \in T_\gamma \Gamma_L$, there exists a section of $b \in \langle B \rangle_U$, defined on a neighbourhood U of $s(\gamma)$ such that $u = \overrightarrow{b}_{|_\gamma}$. Now, Eq. (4) gives
$$L_{\overrightarrow{b}} \pi_\Gamma = [\![\pi_\Gamma, \overrightarrow{b}]\!]_{T\Gamma} = (-1)^k \overrightarrow{\delta(b)}$$
Since $\delta(b)$ is a section of $\langle B \rangle_U$ according to Proposition 3.5, we have again $\overrightarrow{\delta(b)}_{|_\gamma}(\beta) = 0$ for any $\beta \in \wedge^k (\ker(\mathrm{d}_\gamma p))^\perp$ and Condition 2 in Lemma 3.8 is satisfied.

By Lemma 3.8 therefore, the quotient space $Z(R_0) = R_0 \backslash \Gamma_L$ inherits a k-vector field $\pi_{Z(R_0)}$ that satisfies Eq. (7). Our next goal is to show that the triple $(Z(R_0), \phi_0, \pi_{Z(R_0)})$ is a resolution compatible with π_M. In short, we have to show that
$$(\phi_0)_* \pi_{Z(R_0)} = \pi_M. \tag{10}$$

Recall, from 3, that $(-1)^{k-1} t_* \pi_\Gamma = \pi_M$, so that $(-1)^{k-1} \pi_\Gamma [t^* f_1, \cdots, t^* f_k] = t^* \pi_M [f_1, \cdots, f_k]$. Now, Let $f_1 \cdots f_k \in \mathcal{F}(M)$ be local functions. For $i = 1, \ldots, k$, the restriction to Γ_L of the

function $t^* f_i$ is equal to the pull-back through p of the function $\phi_0^* f_i$. In equation:

$$\begin{cases} p^*(\phi_0^* f_1) & = & \imath^*(t^* f_1) \\ & \vdots & \\ p^*(\phi_0^* f_k) & = & \imath^*(t^* f_k) \end{cases}$$

Hence Lemma 3.8 (more precisely Eq. (8)) yields

$$(-1)^{k-1} p^* \pi_{Z(R_0)}[\phi_0^* f_1, \cdots, \phi_0^* f_k] = \pi_\Gamma \left[t^* f_1, \cdots, t^* f_k\right]_{|\Gamma_L} \tag{11}$$

According to Eq. (3), the relation $\pi_\Gamma\left[t^* f_1, \cdots, t^* f_k\right]_{|\Gamma_L} = (-1)^{k-1} t^*(\pi_M[f_1, \cdots, f_k])$ holds. Together with Eq. (11), this implies in turn:

$$\begin{aligned} (-1)^{k-1} p^* \pi_{Z(R_0)}[\phi_0^* f_1, \cdots, \phi_0^* f_k] &= (-1)^{k-1} t^* \pi_M[f_1, \cdots, f_k] \\ &= (-1)^{k-1} p^*(\phi_0^* \pi_M[f_1, \cdots, f_k]). \end{aligned}$$

Since p is a surjective submersion, this amounts to

$$\pi_{Z(R_0)}[\phi_0^* f_1, \cdots, \phi_0^* f_k] = p^* \circ \phi_0^* \pi_M[f_1, \cdots, f_k],$$

and this completes the proof of the relation (10). In conclusion $(Z(R_0), \phi_0)$ is a étale resolution compatible with π_M.

Now, the natural inclusion $R_0 \subset R$, where $R \rightrightarrows L$ is a sub-Lie groupoid of $\Gamma \rightrightarrows M$ integrating the normalisation $B \to L$ of the algebroid crossing L induces a morphism of étale resolution

$$Z(R_0) = R_0 \backslash \Gamma_L \to R \backslash \Gamma_L = Z(R),$$

which, moreover, is a local diffeomorphism. According to Lemma 3.3, the k-vector field $\pi_{Z(R_0)}$ goes to the quotient and defines a k-vector field $\pi_{Z(R)}$ on $Z(R)$ which satisfies itself $\phi_* \pi_{Z(R)} = \pi_M$. This completes the proof. □

Remark 3.9. It is natural to ask what happens for $k = 1$. For a multiplicative vector field X on $\Gamma \rightrightarrows M$, the vector field $s_* X$ is well defined but does not need to be tangent to $\mathcal{S}$. But if we assume that it is tangent to $\mathcal{S}$, then to require an algebroid crossing to be coisotropic with respect to $s_* X$ means that $s_* X$ must be also tangent to L. Under these conditions, one can define by the same method a vector field on $Z(R)$ that projects onto $s_* X$.

For $k = 0$, the situation is as follows. A multiplicative function on $\Gamma \rightrightarrows M$ is simply a function that satisfies $f(\gamma_1\gamma_2) = f(\gamma_1) + f(\gamma_2)$ for all compatible $\gamma_1, \gamma_2 \in \Gamma$. Such a function does not induce any particular object on M. However, when f vanishes on R, it goes to the quotient to yield a function on $Z(R)$.

We finish this section with a proof of Lemma 3.8.

Proof. Choose arbitrary local functions $f_1, \cdots, f_k$ defined on an open subset $U \subset P$. In a neighbourhood $V \subset Q$ of any $m \in \phi^{-1}(m)$, there exists functions $\tilde{f}_1, \cdots, \tilde{f}_k$ such that:

$$\begin{cases} \phi^* f_1 & = & \imath^* \tilde{f}_1 \\ & \vdots & \\ \phi^* f_k & = & \imath^* \tilde{f}_k \end{cases}$$

We want to define a k-vector field π_P on P by

$$(\pi_P)_{|p}[f_1, \cdots, f_k] = (\pi_Q)_{|x}[\tilde{f}_1, \cdots, \tilde{f}_k],$$

where $x \in N$ is any point with $p(x) = p$. We have to check that this definition makes sense: for this purpose it suffices to check that *(i)* the restriction to N of $\pi_Q[\tilde{f}_1, \cdots, \tilde{f}_k]$ does not depend on the choice of the local functions $\tilde{f}_1, \cdots, \tilde{f}_k$, and *(ii)* that the restriction to N of $\pi_Q[\tilde{f}_1, \cdots, \tilde{f}_k]$ is constant along the fibers of $\Phi : N \to P$ (i.e. does not depend on x).

First, we check *(i)*. If we assume that $\tilde{f}_k$ vanishes on $N \cap V$, then we have

$$\mathrm{d}_x\tilde{f}_1 \wedge \cdots \wedge \mathrm{d}_x\tilde{f}_k \in \wedge^{k-1}(\ker(\mathrm{d}_x\Phi))^\perp \wedge T_xN^\perp.$$

As a consequence, the restriction to N of the function $\pi_Q[\tilde{f}_1, \cdots, \tilde{f}_k]$ vanishes; this proves *(i)*.

Now, we prove *(ii)*. The fibers of $\Phi : N \to P$ being connected subsets, it suffices to check that for any $x \in V \cap p^{-1}(U)$ and $u \in T_xN$, we have $u[\pi_Q[\tilde{f}_1, \cdots, \tilde{f}_k]] = 0$. By assumption, there exists a local vector field X through u tangent to N and which satisfies $(L_X\pi_Q)_{|x} \in \wedge^k(\ker(\mathrm{d}_x\Phi))^\perp \wedge T_xN^\perp$. Hence

$$\begin{aligned} u\left[\pi_Q\left[\tilde{f}_1, \cdots, \tilde{f}_k\right]\right]_{|x} &= X\left[\pi_Q\left[\tilde{f}_1, \cdots, \tilde{f}_k\right]\right]_{|x} \\ &= (L_X\pi_Q)_{|x}\left[\tilde{f}_1, \cdots, \tilde{f}_k\right] - \textstyle\sum_{i=1}^k (\pi_Q)_{|x}\left[\tilde{f}_1, \cdots, X[\tilde{f}_i], \cdots, \tilde{f}_k\right]. \end{aligned}$$

But the function $(L_X\pi_Q)[\tilde{f}_1, \cdots, \tilde{f}_k]$ vanishes at the point x by assumption since

$$\mathrm{d}_x\tilde{f}_1, \cdots, \mathrm{d}_x\tilde{f}_k \in (\ker \mathrm{d}_x\Phi)^\perp.$$

Moreover, the restriction to N of $X[\tilde{f}_i]$ vanishes, hence, according to *(i)*, we have

$$(\pi_Q)_{|x}\left[\tilde{f}_1, \cdots, X[\tilde{f}_i], \cdots \wedge \tilde{f}_k\right] = 0, \quad \forall i = 1, \cdots, k.$$

Hence $u\big[\pi_Q[\tilde{f}_1, \cdots, \tilde{f}_k]\big] = 0$ (at the point x). This completes the proof of *(ii)*, and the proof of Lemma 3.8 as well. $\square$

4 Symplectic groupoids and symplectic resolutions.

4.1 Definition of a symplectic resolution.

Let (M, π_M) be a Poisson manifold, and $(T^*M \to M, [\cdot,\cdot]^{\pi_M}, \pi_M^\#)$ the Lie algebroid associated with. By classical theory of Poisson manifold, the leaves of this algebroid are symplectic manifolds, and the inclusion maps are Poisson maps. For a given leaf $\mathcal{S}$, we denote by $\omega_\mathcal{S}$ its symplectic form. By construction, the bivector field π_M on the manifold M is tangent to all the symplectic leaves.

We explain what we mean by a symplectic resolution.

Definition 4.1. *Let $\mathcal{S}$ be a locally closed symplectic leaf, and let $\bar{\mathcal{S}}$ be its closure,* a symplectic resolution of $\bar{\mathcal{S}}$ *is a triple (Z, ω_Z, ϕ) where*

1. *(Z, ϕ) is a resolution of $\bar{\mathcal{S}}$,*

2. *the 2-form ω_Z defined on $\phi^{-1}(\mathcal{S})$ by $\omega_Z := \phi^* \omega_{\mathcal{S}}$ extends to a holomorphic/smooth symplectic 2-form on Z.*

When (Z, ϕ) is only an étale/covering resolution, then we speak of an étale/covering symplectic resolution. *For convenience, we often denote a symplectic resolution as a triple $(Z, \phi \omega_Z)$.*

Remark 4.2. We leave it to the reader to check that an étale symplectic resolution resolution (Z, ϕ, ω_Z) is in particular an étale resolution compatible with π_M, and, indeed, an étale Poisson resolution.

Let us enumerate for clarity all the conditions required in order to have a symplectic resolution (Z, ω_Z, ϕ) of the closure of a symplectic leaf $\mathcal{S}$.

1. Z is a manifold and ω_Z is a symplectic form with Poisson bivector π_Z,

2. $\phi : Z \to M$ is a holomorphic/smooth map from the manifold Z to the manifold M,

3. $\phi(Z) = \bar{\mathcal{S}}$,

4. $\phi^{-1}(\mathcal{S})$ is dense in Z,

5. the restriction of ϕ to a map from $\phi^{-1}(\mathcal{S})$ to $\mathcal{S}$ is a biholomorphism/diffeomorphism,

6. $\phi : Z \to M$ is a Poisson map.

For étale/covering symplectic resolutions, the fifth point above needs to be replaced by "the restriction of ϕ to $\phi^{-1}(\mathcal{S})$ is an étale/covering map over $\mathcal{S}$".

We finish this introductory section by defining complete symplectic resolutions. Let (M_i, π_i), $i = 1, 2$ be Poisson manifolds. Recall that a Poisson map $\phi : M_1 \to M_2$ is said to be *complete* if, for all open subset $U \subset M_2$ and all $f \in \mathcal{F}(U)$, the flow starting at $m \in M_1$ of the Hamiltonian vector field $\mathcal{X}_{\phi^* f}$ is defined at the time $t = 1$ as soon as the flow starting at $\phi(m) \in M_2$ of the Hamiltonian vector field $\mathcal{X}_f$ is defined at the time $t = 1$.

Definition 4.3. *Let (M, π_M) be a Poisson manifold and $\mathcal{S}$ a symplectic leaf. An étale symplectic resolution (Z, ϕ, ω_Z) of $\overline{\mathcal{S}}$ is said to be complete if the map ϕ is a complete Poisson map from (Z, ω_Z^{-1}) to (M, π_M).*

4.2 Symplectic groupoid and symplectic resolution of the closure of a symplectic leaf.

Let (M, π_M) be a Poisson manifold and $\mathcal{S}$ a locally closed symplectic leaf. We introduce in this section the notion of Lagrangian crossing of $\overline{\mathcal{S}}$, which is a particular case of algebroid crossing adapted to the construction of symplectic resolutions. We then specialise Theorem 3.7 to the case of Lagrangian crossing of Poisson manifolds.

Definition 4.4. *Let $\mathcal{S}$ be a symplectic leaf of a Poisson manifold (M, π_M). A* Lagrangian crossing *of $\overline{\mathcal{S}}$ is a submanifold L of M such that*

1. *$L \cap \mathcal{S}$ is dense in L and is a Lagrangian submanifold of $\mathcal{S}$*

2. *L has a non-empty intersection with all the symplectic leaves contained in $\bar{\mathcal{S}}$.*

We immediately connect this notion to the notion of algebroid crossing.

Proposition 4.5. *Any Lagrangian crossing L is an algebroid crossing of $(T^*M \to M, \pi_M^\#, [\cdot,\cdot]^{\pi_M})$ with normalisation $TL^\perp \to L$ and is coisotropic with respect to the bivector field π_M.*

Proof. First, $L \cap \mathcal{S}$ being a Lagrangian submanifold of $\mathcal{S}$, the identity $\pi_M^\#(T_x^* L^\perp) = T_x L$ holds for any $x \in L \cap \mathcal{S}$. By density of $L \cap \mathcal{S}$ in L, the inclusion $\pi_M^\#(T_x^* L^\perp) \subset T_x L$ holds for any $x \in L$. As a consequence, L is coisotropic with respect to π_M. The vector bundle $TL^\perp \to L$ is a is the normalisation of L.

According to Definition 4.4 (2), L intersects all the algebroid leaves contained in $\bar{\mathcal{S}}$, since these leaves are precisely the symplectic leaves. As a consequence, first, L is an algebroid crossing of $\bar{\mathcal{S}}$, and, second, its normalisation is $TL^\perp \to L$. □

In view of Proposition 4.5, one can apply Theorem 3.7 to the particular cases where algebroid crossings with normalisation are Lagrangian crossing of Poisson manifolds.

Theorem 4.6. *Let (M, π_M) be a Poisson manifold, $\mathcal{S}$ be a locally closed symplectic leaf, and L a Lagrangian crossing of $\bar{\mathcal{S}}$. If,*

1. *there exists a symplectic Hausdorff Lie groupoid $(\Gamma \rightrightarrows M, \omega_\Gamma)$ integrating the Poisson manifold M, and*

2. *there exists a closed sub-Lie groupoid $R \rightrightarrows L$ of $\Gamma \rightrightarrows N$, closed as a subset of Γ_L^L, integrating $TL^\perp \to L$,*

then

1. *$(Z(R), \omega_{Z(R)}, \phi)$ is an étale complete symplectic resolution of $\bar{\mathcal{S}}$, where*
   (a) *$Z(R) = R\backslash\Gamma_L$, and,*
   (b) *$\phi : Z(R) \to M$ is the unique holomorphic/smooth map such that the following diagram commutes*

$$\begin{tikzcd} \Gamma_L \arrow[r, "p"] \arrow[dr, "t"'] & Z(R) \arrow[d, "\phi"] \\ & M \end{tikzcd} \tag{12}$$

   where $p : \Gamma_L \to Z(L) = R\backslash\Gamma_L$ is the natural projection, and
   (c) *$\omega_{Z(R)}$ is the symplectic form defined by*

$$p^*\omega_{Z(R)} = -\imath^*\omega_\Gamma. \tag{13}$$

2. *When $L \cap \mathcal{S}$ is a R-connected set, this étale complete symplectic resolution is a covering complete symplectic resolution with typical fiber $\frac{\pi_0(I_x(\Gamma))}{\pi_0(I_x(R))}$, where $x \in \mathcal{S}$ is an arbitrary point, and $I_x(\Gamma)$ (resp. $I_x(R)$) stands for the isotropy group of $\Gamma \rightrightarrows M$ (resp. of $R \rightrightarrows L$) at the point x.*

3. *This étale complete symplectic resolution is a complete symplectic resolution if and only if R contains $\Gamma_{L\cap\mathcal{S}}^{L\cap\mathcal{S}}$. In this case, we have $R = \overline{\Gamma_{L\cap\mathcal{S}}^{L\cap\mathcal{S}}} \cap \Gamma_L^L$.*

4. *When $L\cap\mathcal{S}$ is a connected set and $R \rightrightarrows L$ is the source-connected sub-Lie groupoid of $\Gamma \rightrightarrows M$ with Lie algebroid $TL^\perp \to L$, then the typical fiber is isomorphic to $\frac{\pi_1(\mathcal{S})}{j(\pi_1(L\cap\mathcal{S}))}$, where $j : \pi_1(L\cap\mathcal{S}) \to \pi_1(\mathcal{S})$ is the map induced at the fundamental group level by the inclusion of $L\cap\mathcal{S}$ into $\mathcal{S}$.*

We need, first, a few lemmas.

Lemma 4.7. *For any symplectic groupoid $(\Gamma \rightrightarrows M, \omega_\Gamma)$ that integrates a Poisson manifold (M, π_M), the source map (resp. the target map) is a complete map from $\Gamma = \pi_\Gamma = \omega_\Gamma^{-1}$ to (m, π_M) (resp. $(M, -\pi_M)$).*

Proof. For the real case, we refer to [2]. In the complex case, note, to start with, that for any holomorphic Poisson manifold (N, π_N) where π_N is a holomorphic Poisson structure with real part π_N^R, and any holomorphic local function f, the Hamiltonian vector field $\mathcal{X}_f$ is twice the Hamiltonian vector field (with respect π_N^R) of the real part of f. As a consequence a Poisson map between holomorphic Poisson manifolds is complete if the induced Poisson map between their real parts is complete.

Now, according to [17], the real part of π_Γ is precisely the Poisson bivector field associated to the symplectic structure integrating the real part of π_M. The source and target maps are then complete maps since Lemma 4.7 holds true in the real case. ☐

Lemma 4.8. *Let P_1, P_2 be two Poisson manifolds and $\phi : P_2 \to P_1$ be a Poisson or anti-Poisson map which is a surjective submersion. If L is a coisotropic submanifold in P_1, then $\phi^{-1}(L)$ is coisotropic in P_2.*

Proof. We assume that ϕ is a Poisson map, the anti-Poisson case being similar, up to a sign. Since ϕ is a submersion, for any $x \in P_2$, the dual $(\ker(\mathrm{d}_x\phi))^\perp$ of the kernel of $d_x\phi$ is generated by covectors of the forms $\mathrm{d}_{x_2}(\phi^* f)$, with $f \in \mathcal{F}(P_1)$. Since $\phi : P_2 \to P_1$ is a Poisson map

$$\phi_*(\pi_{P_2}^\#(\mathrm{d}_{x_2}\phi^* f)) = \phi_*(\mathcal{X}_{\phi^* f}(x_2)) = \mathcal{X}_f(\phi(x_2))$$

Since L is coisotropic, the relation $\mathcal{X}_f(\phi(x_2)) \in T_{\phi(x_1)}L$ holds. Since ϕ is a submersion, we obtain $\pi_{P_2}^\#(\mathrm{d}_{x_2}\phi^* f) \in T_{x_2}(\phi^{-1}(L))$. This completes the proof. ☐

Now we turn our attention to the proof of Theorem 4.6.

Proof. First, Conditions 1 and 2 in Theorem 4.6 imply that Conditions 1, 2 and 3 in Theorem 3.7 are satisfied. According to Theorem 3.7 therefore, there exists a bivector field $\pi_{Z(R)}$ on $Z(R)$ such that $(Z(R), \phi, \pi_{Z(R)})$ is an étale resolution of $\bar{\mathcal{S}}$ which is compatible with the bivector field π_M. In view of Definition 4.1, what remains to prove is that, indeed: *(i)* the bivector field $\pi_{Z(R)}$ is the Poisson bivector field associated to a symplectic structure $\omega_{Z(R)}$ on $Z(R)$ that satisfies Eq. (13), and *(ii)* that ϕ is a complete map.

We prove *(i)*. Denote by π_Γ the multiplicative bivector field on Γ associated with the symplectic structure ω_Γ. The submanifold L being coisotropic in M and the source map being an Poisson map, Γ_L is a coisotropic submanifold of the symplectic manifold (Γ, ω_Γ).

Since the 2-differential associated with π_Γ is the de Rham differential (see [15] for instance), Eq. (4) amounts to the following relation

$$\overrightarrow{\mathrm{d}f} = [\![\pi_\Gamma, s^* f]\!]_{T\Gamma} = \mathcal{X}_{s^* f} = \pi_\Gamma^\#(s^* \mathrm{d}f) \tag{14}$$

for any local function $f \in \mathcal{F}(M)$ (where $\mathcal{X}_f$ stands for the Hamiltonian vector field of f). One can immediately rewrite Eq. (14) as

$$\overrightarrow{\alpha} = (\pi_\Gamma^\#)_\gamma(s^*\alpha) \tag{15}$$

for all $\alpha \in T_m\Gamma$ and $\gamma \in \Gamma_m$. Since s^* is a diffeomorphism from $(T_mL)^\perp$ to $(T_\gamma\Gamma_L)^\perp$, Eq. (15) gives:

$$\left\{\overrightarrow{\alpha}_{|\gamma}, \alpha \in T_mL^\perp\right\} = ((\pi^\#_{Z(R)})_{|\gamma} T_\gamma\Gamma_L)^\perp$$

Now, the kernel of the projection map $p : \Gamma_L \to Z(R) = R\backslash\Gamma_L$, at a point $\gamma \in \Gamma_L$, consists precisely of the left-hand term in the previous expression, so that we have:

$$((\pi_\Gamma^\#)_{|\gamma})^{-1}(\ker(\mathrm{d}_\gamma p)) = (T_\gamma\Gamma_L)^\perp \tag{16}$$

Now, for any $z \in Z(R)$ and any $\alpha \in T_z^*Z(R)$, it follows by classical bilinear algebra from Eq. (7) that the tangent vector $(\pi^\#_{Z(R)})_{|z}(\alpha)$ is by construction given by

$$(\pi^\#_{Z(R)})_{|z}(\alpha) = -(\mathrm{d}_\gamma p)\big((\pi_\Gamma^\#)_{|\gamma}(\widetilde{p^*\alpha})\big), \tag{17}$$

where γ is a point in $p^{-1}(z)$ and, $\widetilde{p^*\alpha} \in T_\gamma^*\Gamma$ is a covector whose restriction to $T_\gamma\Gamma_L$ coincides with $p^*\alpha$. In particular, by Eq. (16), if $(\pi^\#_{Z(R)})_{|z}(\alpha) = 0$, then $(\widetilde{p^*\alpha})$ belongs to $(T_\gamma\Gamma_L)^\perp$, and α needs to vanish. In other words, $(\pi^\#_{Z(R)})_{|z}$ is an injective map, which implies that $\pi_{Z(R)}$ is the Poisson bivector field of a symplectic structure $\omega_{Z(R)}$. Eq. (17) amounts to Eq. (13). This completes the proof of *(i)*.

Next, we prove *(ii)*. Let $U \subset M$ be a open subset, $f \in \mathcal{F}(U)$ a function such that the flow Φ_τ^M starting at $m \in U$ is defined for the time $\tau = 1$, and let $z \in \phi^{-1}(m)$ be a point. The target map t from $(\Gamma, -\pi_\Gamma)$ to (M, π_M) is a complete Poisson map, so that the flow Φ_τ^Γ of t^*f starting at γ is defined for $\tau = 1$, where $\gamma \in \Gamma$ is any point such that $p(\gamma) = z$. Since the Hamiltonian vector field $\mathcal{X}_{t^*f}$ is tangent to Γ_L and since $p_*\mathcal{X}_{t^*f} = \mathcal{X}_{\phi^*f}$, the flow starting at z of $\mathcal{X}_{\phi^*f}$ is equal to $p \circ \Phi_\tau^\Gamma$. In particular, it is defined for $\tau = 1$. This completes the proof of *(ii)*. $\square$

Remark 4.9. We are redevable to Jiang-Hua Lu for the following remark. If $R \rightrightarrows L$ is a source-connected Lie groupoid, then, by construction, the procedure that we have used to build the symplectic structure on $Z(R)$ out of the symplectic structure of Γ matches exactly the procedure called symplectic reduction [20] with respect to the coisotropic submanifold Γ_L.

Proposition 2.14 can then be adapted easily.

Proposition 4.10. *Let L_i, $i = 1, 2$ be two Lagrangian crossing of $\bar{\mathcal{S}}$ such that $R_i = \overline{\Gamma^{L_i\cap\mathcal{S}}_{L_i\cap\mathcal{S}}} \cap \Gamma^{L_i}_{L_i}$ is a sub-Lie groupoid of $\Gamma \rightrightarrows M$. Let Z_i, $i = 1, 2$ be the resolutions corresponding to by Theorem 4.6(3), i.e. $Z_i = R_i\backslash\Gamma_{L_i}$. The following are equivalent:*

(i) the symplectic resolutions $(Z_1, \phi_1, \omega_{Z_1})$ and $(Z_2, \phi_2, \omega_{Z_2})$ are isomorphic,

(ii) $\overline{\Gamma^{L_2\cap\mathcal{S}}_{L_1\cap\mathcal{S}}} \cap \Gamma^{L_2}_{L_1}$ is a Lagrangian submanifold of Γ, and the restrictions to this submanifold of the source and the target maps are surjective submersions onto L_1 and L_2 respectively,

(iii) there exists a submanifold I of Γ that gives a Morita equivalence between the Lie groupoids $R_1 \rightrightarrows L_1$ and $R_2 \rightrightarrows L_2$:

$$\begin{tikzcd} \Gamma_{L_1} \arrow[dr] & R_1 \arrow[d, shift left] \arrow[d, shift right] & I \arrow[dl, "s"] \arrow[dr, "t"'] & R_2 \arrow[d, shift left] \arrow[d, shift right] & \Gamma_{L_2} \arrow[dl] \\ & L_1 & & L_2 & \end{tikzcd}$$

In this case moreover, the R_1-module Γ_{L_1} corresponds to the R_2-module Γ_{L_2} with respect to the Morita equivalence I. Also, the submanifold $I \subset \Gamma$ is Lagrangian in Γ.

Proof. Two symplectic resolutions isomorphic as resolutions are isomorphic as symplectic resolutions. The result is just then an immediate consequence of Proposition 2.14.

The only difficulty is to prove the last assertion. The manifold $\Gamma^{L_2\cap\mathcal{S}}_{L_1\cap\mathcal{S}}$ is coisotropic by Lemma 4.8 and is therefore, indeed, Lagrangian because its dimension is half the dimension of Γ. Hence $I = \overline{\Gamma^{L_2\cap\mathcal{S}}_{L_1\cap\mathcal{S}}} \cap \Gamma^{L_2}_{L_1}$ is a Lagrangian submanifold. $\square$

5 Characterisation of symplectic resolutions of the previous type.

We restrict our attention, in this section, to the most interesting case, id est, the case of symplectic resolutions. The aim of the present section is to characterise proper symplectic resolutions of $\overline{\mathcal{S}}$ isomorphic to the symplectic resolutions of the form $(Z(R) = R\backslash\Gamma_L, \phi, \omega_{Z(R)})$ constructed out of a sub-Lie groupoid $R \rightrightarrows L$ integrating a Lie algebroid crossing L as in Theorem 4.6 (3).

Assume that we have a Lagrangian crossing L of the closure $\overline{\mathcal{S}}$ of a locally closed symplectic leaf $\mathcal{S}$ of an integrable complex or real Poisson manifold (M, π_M).

Definition 5.1. *Let L be a Lagrangian crossing of the closure $\bar{\mathcal{S}}$ of a locally closed symplectic leaf $\mathcal{S}$ of a Poisson manifold (M, π_M). A symplectic resolution (Z, ϕ, ω_Z) of $\bar{\mathcal{S}}$ is said to be* L-compatible *if there exists a submanifold L_Z of Z such that the restriction of ϕ to L_Z is a biholomorphism/diffeomorphism onto L.*

Example 5.2. According to Lemma 2.15, the symplectic resolution $(Z(R) = R\backslash\Gamma_L, \phi_{Z(R)}, \omega_{Z(R)})$ constructed as in Theorem 4.6 (3) is L-compatible. In this case, we have $L_Z = j(L)$.

Remark 5.3. For any L-compatible symplectic resolution, we have $L_Z = \overline{\phi_Z^{-1}(L \cap \mathcal{S})}$. In particular, the manifold L_Z that appears in Definition 5.1 is unique. Indeed, we could characterise L-compatible symplectic resolution as being those such that $\overline{\phi_Z^{-1}(L \cap \mathcal{S})}$ is a submanifold of Z to which the restriction of ϕ is a biholomorphism/diffeomorphism onto L.

A resolution (Z, ϕ) is said to be *proper* if the map ϕ is a proper map. Note that a proper symplectic resolution is always complete.

Example 5.4. The symplectic resolution constructed in Proposition 6.1 is not proper, while the Springer resolution of a Richardson orbit is.

Theorem 5.5. *Let L be a Lagrangian crossing of $\overline{\mathcal{S}}$, where $\mathcal{S}$ is a locally closed symplectic leaf $\mathcal{S}$ of a Poisson manifold (M, π_M).*

Let (Z, ϕ_Z, ω_Z) be a L-compatible complete symplectic resolution of $\overline{\mathcal{S}}$.

Let $\Gamma \rightrightarrows M$ be a source-simply connected and source-connected symplectic groupoid that integrates the Poisson manifold (M, π_M). Then:

1. *there exists a sub-Lie groupoid $R \rightrightarrows L$ of $\Gamma \rightrightarrows M$ integrating $TL^{\perp} \to L$ closed in Γ_L^L and containing $\Gamma_{L\cap\mathcal{S}}^{L\cap\mathcal{S}}$;*
2. *the symplectic resolution $(Z(R) = R\backslash\Gamma_L, \phi_{Z(R)}, \omega_{Z(R)})$ (whose existence is granted by Theorem 4.6 (3)) is isomorphic (as a symplectic resolution) to an open subset of (Z, ϕ_Z, ω_Z);*
3. *if, moreover, the symplectic resolution $(Z(R) = R\backslash\Gamma_L, \phi_{Z(R)}, \omega_{Z(R)})$ is proper, then the symplectic resolutions (Z, ϕ_Z, ω_Z) and $(Z(R) = R\backslash\Gamma_L, \phi_{Z(R)}, \omega_{Z(R)})$ are isomorphic (as symplectic resolutions).*

Before proving Theorem 5.5, we have to adapt in our context a result from the proof of Theorem 8 in [4] and to state that (Z, ϕ_Z) is a right Γ-module. There are important differences between our case and the setting of [4]. The authors of [4] work with symplectic realizations, id est, symplectic varieties (S, ω_S) endowed with a surjective submersion from S to M which is also a Poisson map, while we do not assume ϕ to be surjective in general. Moreover, the holomorphic case is not considered in their work. Also, we prefer to work with right action, while [4] works with left action, but this last point makes of course no major difference. However, the following fact, adapted from the proof of Theorem 8 in [4], remains valid.

Proposition 5.6. *Let $(\Gamma \rightrightarrows M\omega_\Gamma)$ be a source-connected and source-simply connected symplectic groupoid integrating a Poisson manifold (M, π_M). Let $\mathcal{S}$ be a locally closed symplectic leaf, and (Z, ϕ_Z) an étale symplectic resolution of $\overline{\mathcal{S}}$. There is a unique action of the Lie groupoid $\Gamma \rightrightarrows M$ on $Z \stackrel{\phi_Z}{\to} M$ whose restriction to $\phi_Z^{-1}(\mathcal{S})$ is given by*

$$\gamma \cdot z = \phi_Z^{-1}(t(\gamma)) \quad \forall z \in \phi_Z^{-1}(\mathcal{S}), \gamma \in \Gamma_{s(z)}. \tag{18}$$

Example 5.7. When $(Z = R\backslash\Gamma_L, \phi_Z)$ is a symplectic resolution associated to a sub-Lie groupoid $R \rightrightarrows M$ integrating a Lagrangian crossing L as in Theorem 4.6(3), then the unique action that satisfies Equation (18) is the action induced by the right action of Γ to itself. More precisely, it is given by $\gamma \cdot [g] = [g\gamma]$ for all $[g] \in R\backslash\Gamma_L, \gamma \in \Gamma$ with $s(\gamma) = \phi_Z([g])$, where $[g'] \in R\backslash\Gamma_L$ stands for the class in $R\backslash\Gamma_L$ of a given element $g' \in \Gamma_L$.

Proof. Since ϕ_Z is an isomorphism from $\phi_Z^{-1}(\mathcal{S})$ to $\mathcal{S}$, there is at most one Lie groupoid action of $\Gamma \rightrightarrows M$ on $Z \stackrel{\phi_Z}{\to} M$ that satisfies Eq. (18) by density of $\phi_Z^{-1}(\mathcal{S})$ in Z. This proves uniqueness.

We sketch the argument of the existence and prove in detail only those which differ from [4].

Recall that a *cotangent path* is a map $a(u)$ of class $\mathcal{C}^1$ from $[0,1]$ to T^*M satisfying

$$\frac{\mathrm{d}m(u)}{\mathrm{d}u} = \pi_M^\#\big(a(u)\big) \quad \forall u \in [0,1]$$

where $m(u)$ is the base path of $a(u)$, id est, the projection of $a(u)$ onto M (through the canonical projection $T^*M \to M$). Also, one assumes that $u \mapsto a(u)$ is equal to zero for all u in neighbourhoods of 0 and 1, so that one can concatenate two A-paths when the end point of the first one coincides with the starting point of the second one. There is a notion of homotopy of cotangent paths, (see [4]), and it is now a classical result that if the Poisson manifold (M, π_M) integrates to a source-connected and source-simply connected groupoid $\Gamma \rightrightarrows M$, there is an isomorphism

$$\Gamma \simeq \frac{\text{cotangent paths}}{\text{homotopy}}. \tag{19}$$

The source and targets of the element $\gamma \in \Gamma$ corresponding to a cotangent path are the starting and end points of its base path respectively. Product in Γ corresponds to concatenation of cotangent paths.

The argument used in the step 2 of the proof of theorem 8 in [4] does not use the assumption that what the map denoted by μ in [4] (and which is our ϕ_Z) is a surjective submersion. It therefore remains valid and yield the following result: given a cotangent path $a(u)$ with base path $m(u)$, and some $z \in Z$ with $\phi_Z(z) = m(1)$, there exists a unique curve $z(u)$ on Z with starting point $z(0) = z$ and which satisfies the differential equation

$$\begin{cases} \frac{\mathrm{d}z(u)}{\mathrm{d}u} & = & \pi_Z^\#\big(\phi_\Sigma^*(a(u)\big) & \\ \phi_\Sigma\big(\sigma(u)\big) & = & m(u) & \forall u \in [0,1] \end{cases} \tag{20}$$

Lemma 2 in [4] does not use the fact that the map called μ in [4] is a surjective submersion and remains valid. Its conclusion is that $z(1)$ does not depend on the class of homotopy of the cotangent path $a(u)$. This second point, together with the identification given in Eq. (19), yields the existence of a map $Z \times_{\phi_Z, M, s} \Gamma \to Z$. This map defines a (right)-groupoid action (as shown in the end of Step 2 in [4], up to the fact that [4] considers left action). Eq. (18) is automatically satisfied.

The only delicate point is to show that this action is indeed holomorphic in the complex case. But it follows immediately from Eq. (18) that the restriction to $\phi_Z^{-1}(\mathcal{S}) \times_{\phi_Z, M, s} \Gamma_{\mathcal{S}}$ of the action map $Z \times_{\phi_Z, M, s} \Gamma \to Z$ is holomorphic. Since $\phi_Z^{-1}(\mathcal{S})$ is dense is Z, the action map is holomorphic. $\square$

Now, we can turn our attention to the proof of Theorem 5.5.

Proof. 1) We use the shorthand $j = (\phi_Z)_{|Z}$ to denote the restriction of ϕ_Z to a biholomorphism/diffeomorphism from L_Z onto L. The action of $\Gamma \rightrightarrows M$ on $Z \stackrel{\phi_Z}{\to} M$ restricts and yields a map, that we denote by Ξ, from $L_Z \times_{j, L, s} \Gamma_L$ to Z. But $L_Z \times_{j, L, s} \Gamma_L$ is simply isomorphic to Γ_L (the isomorphism being simply the projection onto the second component), so that Ξ can be considered as a map from Γ_L to Z,

The relation (18) can be rewritten as $\phi_Z \circ \Xi = t$. Under this form, it implies that the restriction of Ξ to $\Gamma_{L \cap \mathcal{S}}$ is given by $\phi_Z^{-1} \circ t$. This fact has several consequences.

1. First, $\phi_Z^{-1}(\mathcal{S}) \subset \Xi(\Gamma_L)$.

2. Let ω_{Γ_L} be the restriction of ω_Γ to Γ_L. The following relation holds:

$$\Xi^* \omega_Z = -\omega_{\Gamma_L} \tag{21}$$

   Let us prove this relation. Since the target map is an anti-Poisson map, we have the relation $t^*\omega_{\mathcal{S}} = (-\omega_\Gamma)|_{\Gamma^{\mathcal{S}}}$. But $\Gamma_{L\cap\mathcal{S}} \subset \Gamma^{\mathcal{S}}$, and we can conclude that $t^*\omega_{\mathcal{S}} = -\omega_{\Gamma_L}$. This relation, together with the relation $\Xi^* \circ \phi_Z^* = t^*$ and the fact that phi_Z is a symplectomorphism from $(\phi_Z^{-1}(\mathcal{S}), \omega_Z)$ to $(\mathcal{S}, \omega\mathcal{S})$ implies that Eq. (21) holds on $\phi_Z^{-1}(\mathcal{S})$, hence on Z by density.

3. Let $\varepsilon(L) \subset \Gamma_L$ be the image of L through the unit map. The restriction of Ξ to $\varepsilon(L)$ takes its values in L_Z. Since both $t : \varepsilon(L) \to L$ and j : are biholomorphisms/diffeomorphisms, the restriction of Ξ to $\varepsilon(L)$ is a biholomorphism/diffeomorphism onto L_Z.

Let us show that Ξ is a submersion onto its image. Recall that the rank of ω_{Γ_L} is $dim(\mathcal{S})$, as well as the rank of ω_Z. Eq. (21) implies then that $\mathrm{d}_\gamma\Xi$ is surjective at every point $\gamma \in \Gamma_L$, id. est, that Ξ is a submersion onto its image.

The inverse image of L_Z through Ξ is therefore a sub-manifold of Γ_L that we denote by R. Let us describe all the properties of R. First, by construction, $R \subset \Gamma$ is the set of points in $r \in \Gamma$ such that $\Xi(r, m) \in L_Z$ where $m = j^{-1}(s(r))$. With the help of that characterisation, we leave it to the reader to check that Ξ is stable by inverse. Since Ξ is a submersion onto its image, the restriction of Ξ to R is a submersion onto its image L_Z, and the restriction of the target map t to R is also a submersion onto its image L. Since the inverse map intertwines the source and the target maps, the restriction of the source map s to R is also a submersion onto ts image L. Now, one sees easily that the product of two compatible elements in $R$is in R also. Therefore $R \rightrightarrows L$ is a groupoid. Since it obviously contains $\Gamma_{L\cap\mathcal{S}}^{L\cap\mathcal{S}}$ as a dense open subset, it proves (1).

2) According to Theorem 4.6 (3), $(Z(R) = R\backslash\Gamma_L, \phi_{Z(R)}, \omega_{Z(R)})$ is a symplectic resolution of $\bar{\mathcal{S}}$. The map Ξ goes to the quotient and yields a map, that we denote Ψ from $Z(R)$ to Z. By construction, the restriction of Ψ to the dense open subset $\phi_{Z(R)}^{-1}(\mathcal{S})$ is given by $\Xi = \phi_Z^{-1} \circ \phi_{Z(R)}$. In particular, it is a one-to-one map and it is a symplectomorphism. By density, it implies that Ψ is a symplectomorphism, and is therefore an open map, and a local diffeomorphism onto its image. But a local diffeomorphism which is a diffeomorphism on a dense open subset is a diffeomorphism. This completes the proof.

3) It remains to prove that Ψ is onto when $\phi_{Z(R)}$ is proper. Let U be an open subset of Z contained in $\psi_Z^{-1}(\mathcal{S})$ whose closure $\overline{U}$ (w.r.t. the topology of the manifold Z) is compact. Let $V = \Psi^{-1}(U)$. It is elementary that $V = \phi_{Z(R)}^{-1}(\phi_Z(U))$, so that $V \subset \phi_{Z(R)}^{-1}(\phi_Z(\overline{U}))$. But $\phi_Z(\overline{U})$ is compact since ϕ_Z is continuous and $\phi_{Z(R)}^{-1}(\phi_Z(\overline{U}))$ is compact by properness of $\phi_{Z(R)}$. Now, $\Psi(\overline{V})$ is compact and contains U, it therefore contains $\overline{U}$. Since any point in Z lies inside the closure of a relatively compact open set contained in the dense open subset $\psi_Z^{-1}(\mathcal{S})$, Ψ is surjective. This completes the proof. $\square$

6 Examples

6.1 Poisson brackets on $\mathbb{R}^2$

In this section, we work in the setting of real differential geometry. We present an example for which the space we are working on is a regular manifold, but the Poisson structure has singularities.

Let $\kappa(x, y)$ be a smooth non-negative function on $\mathbb{R}^2$. We assume that the set of zeroes $(z_i)_{i\in I}$ is a discrete subset of $\mathbb{R}^2$; so that we can assume $I \subset \mathbb{N}$. Define a Poisson bracket on $\mathbb{R}^2$ by

$$\{x, y\} = \kappa(x, y)$$

We focus on the symplectic leaf $\mathcal{S} = \mathbb{R}^2 - \{z_i, i \in I\}$. Note that in this case $\bar{\mathcal{S}} = \mathbb{R}^2$ is a smooth manifold.

Any smooth submanifold L of dimension 1, i.e. any smooth embedded curve, is a coisotropic submanifold, whose intersection with $\mathcal{S}$ is Lagrangian. So that any smooth embedded curve L that goes through all the points $z_i, i \in I$ is a Lagrangian crossing. Such a curve always exists.

According to Corollary 5 in [4], this Poisson manifold is integrable and integrates to a Lie groupoid $\Gamma \rightrightarrows \mathbb{R}^2$. Theorem 4.6 allows us to built symplectic étale resolutions whenever $TL^\perp \to L$ integrates to a sub-Lie groupoid of $\Gamma \rightrightarrows M$ closed in Γ_L^L. In a future work, we shall see that this is always the case: more precisely there exists a source-connected sub-Lie groupoid of $\Gamma \rightrightarrows M$ as well as a sub-Lie groupoid containing $\Gamma_{L\cap\mathcal{S}}^{L\cap\mathcal{S}}$ integrating $TL^\perp \to L$ and closed in Γ_L^L. However, we restrict ourself here to the most basic example.

Example: the case of the bracket $\{x, y\} = x^2 + y^2$.

We introduce complex coordinates $z = x + iy$, $\bar{z} = x - iy$, and we study the case $\kappa(x, y) = x^2 + y^2 = z\bar{z}$. According to [6], the symplectic Lie groupoid is given in this case by

1. $\Gamma := \mathbb{C} \times \mathbb{C}$,
2. the source map $s(Z, z) = z$ and the target map $t(Z, z) = e^{Z\bar{z}} z$,
3. the product $(Z_1, z_1) \cdot (Z_2, e^{Z_1\bar{z}_1} z_1) = (Z_1 + e^{\bar{Z}_1 z_1} Z_2, z_1)$
4. the symplectic structure

$$\omega_\Gamma = z\bar{z} dZ \wedge d\bar{Z} + 2\mathcal{R}\mathrm{e}(\overline{z\bar{Z}} dZ \wedge dz) + Z\bar{Z} d\bar{z} \wedge dz + 2\mathcal{R}\mathrm{e}(dZ \wedge d\bar{z}). \qquad (22)$$

(Indeed, the explicit structures of [6] have been slightly modify in order to match our previous conventions). The real axis is a Lagrangian crossing that we denote by L. By construction, $\Gamma_L = \{(Z, \lambda) | Z \in \mathbb{C}, \lambda \in \mathbb{R}\}$. The sub-Lie groupoid $R^{(0)} = \mathbb{R}^2 \subset \Gamma$ integrates $TL^\perp \to L$. So that all the assumptions of Theorem 4.6 are satisfied. We now describe explicitly the symplectic étale resolution obtained by this procedure.

The quotient space $Z(R^{(0)}) = R^{(0)} \backslash \Gamma_L$ is given by

$$Z(R^{(0)}) \simeq \frac{\mathbb{C} \times \mathbb{R}}{\sim}$$

where $(Z,\lambda) \sim (Ze^{M\nu}+M,\nu)$ whenever $\lambda = \nu e^{M\nu}$. In any class of the equivalence relation $\sim$, there is one and only one element of the form (ia,b) with $a,b \in \mathbb{R}$. Let us justify this point. Uniqueness is straightforward. Now, we set $M = -(\mathrm{Im}(Z))e^{-Z\lambda}$ and $\nu = \lambda e^{-Z\lambda}$. One checks easily that $t(M,\nu) = s(Z,\lambda)$ and that $(M,\nu)(Z,\lambda)$ is of the requested form (i.e. the real part of the first component vanishes). As a consequence, $Z(R^{(0)})$ is simply isomorphic to $\mathbb{R}^2$. The target map $t : \Gamma_L \to \mathbb{R}^2 \simeq \mathbb{C}$ factorizes to yield the map

$$\phi(a,b) := be^{iab} = (b\cos(ab), b\sin(ab))$$

and $(Z(R^{(0)}),\phi)$ is an étale resolution of $\overline{\mathcal{S}} \simeq \mathbb{R}^2$. According to Theorem 4.6, $Z(R^{(0)})$ endows a symplectic structure $\omega_{Z(R^{(0)})}$ such that ϕ is a Poisson map. We could compute this structure from the one on Γ as given by (22), but the computation is quite tedious. It is much easier, since we know by Theorem 4.6 that this structure has to exist, to deduce directly from the fact that ϕ is Poisson map the explicit form of the Poisson bracket $\{\cdot,\cdot\}_{Z(R^{(0)})}$ corresponding to $\omega_{Z(R^{(0)})}$. We proceed as follows. The definition of a Poisson map yields the relation

$$\{b\cos(ab), b\sin(ab)\}_{Z(R^{(0)})} = \phi^*\{x,y\}_{\mathbb{R}^2} \tag{23}$$

where x, y are the coordinate functions of $M \simeq \mathbb{R}^2$, and where the relations $\phi^*x = b\cos(ab)$ and $\phi^*y = b\sin(ab)$ have been used. On the one hand, by a direct computation, we obtain

$$\phi^*\{x,y\}_{\mathbb{R}^2} = \phi^*(x^2+y^2) = b^2(\cos^2(ab)+\sin^2(ab)) = b^2 \tag{24}$$

and on the other hand the Leibniz rule gives

$$\{b\cos(ab), b\sin(ab)\}_{Z(R^{(0)})} = b^2\{a,b\}_{Z(R^{(0)})}. \tag{25}$$

According to Equations. (23-24-25), the induced structure on $Z(R^{(0)})$ is simply given by $\{a,b\}_{Z(R^{(0)})} = 1$, which corresponds to the symplectic 2-form

$$\omega_{Z(R^{(0)})} = \mathrm{d}a \wedge \mathrm{d}b$$

In order to find "better" resolutions, one has to replace $R^{(0)} \rightrightarrows L$ by a bigger sub-Lie groupoid. This can be done as follows. Fix $k \in \mathbb{N}$ and let

$$\begin{aligned} R^{(k)} &:= \{(\nu + n\frac{ik\pi}{\lambda},\lambda) \,|\, \nu \in \mathbb{R}, \lambda \in \mathbb{R}^*, n \in \mathbb{Z}\} \cup \{(\nu,0) \,|\, \nu \in \mathbb{R}\} \\ &= R^{(0)} \cup (\cup_{n\in\mathbb{Z}}\{(\nu + n\frac{ik\pi}{\lambda},\lambda) \,|\, \nu \in \mathbb{R}, \lambda \in \mathbb{R}^*, n \in \mathbb{Z}^*\}) \end{aligned}$$

For all $k \in \mathbb{Z}$, $R^{(k)}$ is a closed sub-Lie groupoid of $\Gamma \rightrightarrows M$ with Lie algebroid $TL^\perp \to L$. For $k = 0$, we recover the previous case, for $k = 1$, we have obviously

$$R^{(1)} = \overline{\Gamma^{L\cap\mathcal{S}}_{L\cap\mathcal{S}}} \cap \Gamma^L_L.$$

By Theorem 4.6 (3) therefore, $Z(R^{(1)}) = R^{(1)}\backslash\Gamma_L$ is a symplectic resolution of $\bar{\mathcal{S}}$. The quotient space $Z(R^{(k)}) = R^{(k)}\backslash\Gamma_L$ is given by

$$Z(R^{(k)}) := \frac{\mathbb{C}\times\mathbb{R}}{\sim_k}$$

where $(Z,\lambda) \sim_k \big((-1)^{kn} Z e^{M\nu} + M + \frac{ikn\pi}{\nu}, \nu\big)$ whenever $\lambda = \nu(-1)^{nk} e^{M\nu}$. In any class of the relation $\sim_k$, there is at least one element of the form (ia, b) with $a, b \in \mathbb{R}$, and a short computation shows that $Z(R^{(k)})$ is indeed isomorphic to $\mathbb{R}^2/\sim'_k$, where $\sim'_k$ is the equivalence relation that identifies (a,b) with $((-1)^{kn} a + \frac{nk\pi}{b}, (-1)^{kn} b)$ for all $n \in \mathbb{Z}$ whenever $b \neq 0$.

The symplectic form $\omega_{Z(R^{(0)})} = \mathrm{d}a \wedge \mathrm{d}b$ goes to the quotient through the equivalence relation $\sim'_k$, as one easily checks. This achieves the construction of a symplectic resolution of $\bar{\mathcal{S}}$.

We summarise this construction as follows, that presents the extremal cases $k = 0$ and $k = 1$.

Proposition 6.1. *1. The triple $(Z(R^{(0)}), \phi, \omega_{Z(R^{(0)})})$ is an étale symplectic resolution of $\bar{\mathcal{S}}$, more precisely:*

(a) $Z(R^{(0)}) \simeq \mathbb{R}^2$, we denote (a,b) the canonical coordinates

(b) $\omega_{Z(R^{(0)})} = \mathrm{d}a \wedge \mathrm{d}b$

(c) $\phi(a,b) := b e^{iab} = \big(b\cos(ab), b\sin(ab)\big)$

2. The triple $(Z(R^{(1)}), \phi_1, \omega_{Z(R^{(1)})})$ is a symplectic resolution of $\bar{\mathcal{S}}$, more precisely:

(a) $Z(R^{(1)}) \simeq \mathbb{R}^2/\sim'_1$, where $\sim'_1$ is the equivalence relation that identifies (a,b) with $((-1)^n a + \frac{n\pi}{b}, (-1)^n b)$ for all $n \in \mathbb{N}$ whenever $b \neq 0$. We denote by $\overline{(a,b)} \in Z(R^{(1)})$ the class of $(a,b) \in Z(R^{(0)})$

(b) $\omega_{Z(R^{(1)})}$ is the unique symplectic form satisfying $\Pi^* \omega_{Z(R^{(1)})} = \omega_{Z(R^{(0)})}$ where $\Pi : Z(R^{(0)}) \to Z(R^{(1)})$ is the canonical projection

(c) ϕ_1 is (well)-defined by $\phi_1\big(\overline{(a,b)}\big) = (b\cos(ab), b\sin(ab))$.

3. The projection Π is a morphism of étale symplectic resolutions.

Of course, the previous constructions could be done for any straight line through the origin, since the Poisson structure is invariant under a rotation centred at the origin. Two such resolutions are in general not isomorphic.

Proposition 6.2. *Let L_1 and L_2 be two straight lines through the origin. Let (Z_1, ϕ_1) and (Z_2, ϕ_2) be the two symplectic resolutions associated with as in Proposition 6.1(2). The resolutions (Z_1, ϕ_1) and (Z_2, ϕ_2) are isomorphic if and only if $L_1 = L_2$.*

Proof. By symmetry, one can assume that L_1 is the real axis. Then, by definition of the groupoid structure on $\Gamma \rightrightarrows M$

$$\begin{aligned} \Gamma_{L_1 \cap \mathcal{S}}^{L_2 \cap \mathcal{S}} &= \{\, (Z,\lambda) \,|\, \lambda \in \mathbb{R}^*, \lambda e^{Z\lambda} \in L_2 \,\} \\ &= \{\, (Z,\lambda) \,|\, \lambda \in \mathbb{R}^*, e^{Z\lambda} \in L_2 \,\} \\ &= \{\, (Z,\lambda) \,|\, \lambda \mathrm{Im}(Z) - \alpha \in \pi\mathbb{Z} \,\}, \end{aligned}$$

where $\alpha \in \mathbb{R}$ is the angle with the horizontal line. We leave it to the reader to check that $\overline{\Gamma_{L_1 \cap \mathcal{S}}^{L_2 \cap \mathcal{S}}}$ has an empty intersection with $s^{-1}(0)$ when α does not belong to $\pi\mathbb{Z}$. (In other

words, the latter means that there is no sequences $(\lambda_k)_{k\in\mathbb{N}}, (Z_k)_{k\in\mathbb{N}}$ with λ_k converging to 0 and with Z_k convergent so that $\lambda_k \mathrm{Im}(Z_k) - \alpha \in \pi\mathbb{Z}$ for all $k \in \mathbb{N}$). In conclusion, when L_2 is not the horizontal axis, the restriction of the source map s to $\overline{\Gamma_{L_1\cap\mathcal{S}}^{L_2\cap\mathcal{S}}}$ is not a surjective map onto L_1. According to Proposition 4.10 therefore, the resolutions (Z_1, ϕ_1) and (Z_2, ϕ_2) can not be isomorphic. □

6.2 Nilpotent orbits of a semi-simple Lie algebra: the Springer resolution

We construct, as a particular case of the symplectic resolutions previously built, a very classical resolution called the *Springer resolution*.

In the present case, the space we are working on is a singular variety, but the Poisson structure is "as regular as possible" in the sense that it is symplectic at regular points.

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra and G a connected Lie group integrating $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is identified with its dual $\mathfrak{g}^*$ with the help of the Killing form.

We identify the linear Poisson manifold $\mathfrak{g}^*$ (endowed with the linear Poisson structure) with $\mathfrak{g}$ with the help of the Killing form. This Poisson manifold is integrable. More precisely, it integrates to the transformation Lie groupoid $T^*G \simeq G \times \mathfrak{g} \rightrightarrows \mathfrak{g}$ where

1. the source map, target map and product are as follows

$$\left\{ \begin{array}{ccc} s(g,u) & = & u \\ t(g,u) & = & Ad_{g^{-1}}(u) \\ (g,u)\cdot(h, Ad_{g^{-1}}u) & = & (gh,u) \end{array} \right.$$

2. the symplectic structure is the canonical symplectic structure on a cotangent bundle.

See [2] or [7] for more details.

Let $\mathcal{S}$ a nilpotent orbit in $\mathfrak{g}$. The fundamental assumption that we have to make in order to construct an étale resolution of $\bar{\mathcal{S}}$ is the following. Assume that $\mathcal{S}$ is a *Richardson orbit*, i.e. that there exists a parabolic subalgebra $\mathfrak{P} \subset \mathfrak{g}$ whose nilradical $\mathfrak{U}$ satisfies the property that $\mathfrak{U} \cap \mathcal{S}$ is dense in $\mathfrak{U}$.

Lemma 6.3. *The nilradical $\mathcal{U}$ is a Lagrangian crossing of $\bar{\mathcal{S}}$.*

I am strongly grateful to P. Tauvel and R. Yu for the following proof.

Proof. According to the Theorem of Richardson, (see [23]), for any $x \in \mathfrak{U} \cap \mathcal{S}$, the intersection of the G-orbit of x with $\mathfrak{P}$ is the P-orbit of x. As a consequence, for any $v_1, v_2 \in T_x\mathfrak{U} \simeq \mathfrak{U}$, there exists $p_1, p_2 \in \mathfrak{P}$ such that $v_i = [p_i, x]$, $i = 1, 2$. By the definition of the symplectic structure $\omega_{\mathcal{S}}$ of $\mathcal{S}$, we have

$$\omega_{\mathcal{S}}(v_1, v_2) = \langle x, [p_1, p_2] \rangle = \langle [x, p_1], p_2 \rangle .$$

The spaces $\mathfrak{P}$ and $\mathfrak{U}$ being dual to each other w.r.t. the Killing form, this amounts to $\omega_{\mathcal{S}}(v_1, v_2) = \langle [x, p_1], p_2 \rangle = 0$. In conclusion, $\mathfrak{U}$ is a coisotropic submanifold of $\mathfrak{g}$.

Now, choose some $x \in \mathfrak{U} \cap \mathcal{S}$ and some $w \in T_x\mathcal{S}$. There exists some $a \in \mathfrak{g}$ such that $w = [a, x]$.

Assume that $\omega_{\mathcal{S}}(v, w) = 0$ for all $v \in T_x\mathfrak{U}$. Since $[\mathfrak{P}, x] = \mathfrak{U}$, we have

$$\langle x, [p, a]\rangle = \langle [x, p], a\rangle = 0 \ \forall p \in \mathfrak{P}$$

Hence $a \in \mathfrak{U}^{\perp} = \mathfrak{P}$ and $v \in T_x\mathfrak{U}$. Therefore $\mathfrak{U} \cap \mathcal{S}$ is Lagrangian in $\mathcal{S}$.

The last delicate point is to check that $\mathfrak{U}$ intersects all the symplectic leaves included of $\bar{\mathcal{S}}$.

Let P be the connected parabolic subgroup that integrates $\mathfrak{P}$. Since G/P is projective, the projection Π_2 onto the second component $G/P \times \mathfrak{g} \to \mathfrak{g}$ is a closed map. Now G/P is the Grassmannian of all Lie subalgebras of $\mathfrak{g}$ conjugate to $\mathfrak{P}$. The set S of pairs $(\tilde{\mathfrak{P}}, x) \in G/P \times \mathfrak{g}$ such that x belongs to the nilradical of $\tilde{\mathfrak{P}}$ is a closed subset. Since Π_2 is a closed map, $\Pi_2(S)$ is a closed subset of $\mathfrak{g}$. But $\Pi_2(S)$ is precisely the union of all the adjoint orbits through $\mathfrak{U}$. This completes the proof. $\square$

We have an identification $\mathfrak{P} \simeq \mathfrak{U}^{\perp}$. As a consequence, the sub-Lie groupoids of $G \times \mathfrak{g} \rightrightarrows \mathfrak{g}$ that integrate the subalgebroid $\mathfrak{P} \times \mathfrak{U} \to \mathfrak{U}$ are all the Lie groupoids of the form $P \times \mathfrak{U} \rightrightarrows \mathfrak{U}$ where P is any parabolic Lie subgroup of G with Lie algebra $\mathfrak{P}$. In this case, one can identify $\Gamma_{\mathfrak{U}}$ with $G \times \mathfrak{U}$ and the left action of $P \times \mathfrak{U} \rightrightarrows \mathfrak{U}$ corresponds precisely to the diagonal action of P given by

$$p \cdot (g, u) = (pg, Ad_p u) \quad \forall g \in G, u \in \mathfrak{U}, p \in P.$$

so that $Z_{\mathfrak{U}} = \frac{G \times \mathfrak{U}}{P}$ and $\phi(\overline{(g, u)}) = Ad_{g^{-1}} u$ where $\overline{(g, u)}$ stands for the class of $(g, u) \in G \times \mathfrak{U}$.

Since $\mathfrak{U}$ is a connected set, $\frac{G \times \mathfrak{U}}{P}, \phi)$ is an étale symplectic resolution with typical fiber $\frac{\pi_0(Stab_G(x))}{\pi_0(Stab_P(x))}$. According to Theorem 4.6, it is a symplectic resolution if and only if

$$\pi_0(Stab_P(x)) = \pi_0(Stab_G(x)).$$

We recover as a particular case of Theorem 4.6 the following Proposition, which seems well-known (see Proposition 3.15 in [10] for instance).

Proposition 6.4. *[10] The Springer resolution* $(\frac{G \times \mathfrak{U}}{P}, \overline{(g, u)} \to Ad_{g^{-1}u})$ *of the closure* $\bar{\mathcal{S}}$ *of a Richardson orbit* $\mathcal{S}$ *is a covering symplectic resolution for any parabolic subgroup* P *such that* $\mathfrak{U} \cap \mathcal{S}$ *is dense in* $\mathcal{S}$*, where* $\mathfrak{U}$ *is the nilradical of Lie*(P)*. It is a symplectic resolution if and only if there exists a such a parabolic subgroup* P *such that* $\pi_0(Stab_P(x)) = \pi_0(Stab_G(x))$.

Remark 6.5. For any $g \in G$, and any $\mathfrak{U}$ and P as in Proposition [10] above, $\mathrm{Ad}_g\mathfrak{U}$ is again the nilradical of the Lie algebra of the parabolic subgroup gPg^{-1}. In particular, $\mathrm{Ad}_g\mathfrak{U}$ is again a Lagrangian crossing of $\overline{\mathcal{S}}$ and $\mathrm{Ad}_g\mathfrak{U}$ is a Lie groupoid that integrates it. In particular, one can form a second symplectic resolutions with $Z' := \frac{G \times \mathrm{Ad}_g(\mathfrak{U})}{gPg^{-1}}$ and $\phi' : Z' \mapsto \bar{\mathcal{S}}$ defined as before. This second symplectic resolution is isomorphic to the first one. The Lagrangian closed submanifold I of Γ that gives, according to Proposition 4.10, the Morita equivalence between the Lie groupoids $P \rightrightarrows \mathfrak{U}$ and $gPg^{-1} \rightrightarrows \mathrm{Ad}_g\mathfrak{U}$ is

$$I := gP \times \mathfrak{U} \subset G \times \mathfrak{g}.$$

6.3 Exact multiplicative k-vector fields

For any algebroid $(A \to M, \rho, [\cdot,\cdot])$, and any Lie groupoid $\Gamma \rightrightarrows M$ that integrates it, and any $\Lambda \in \Gamma(\wedge^k A \to M)$, the k-vector field $\overrightarrow{\Lambda} - \overleftarrow{\Lambda}$ is multiplicative. The k-vector fields it defines on M is simply $\rho(\Lambda)$, and is tangent to all algebroid leaves.

We choose a locally closed algebroid leaf $\mathcal{S}$ and an algebroid crossing L of $\overline{\mathcal{S}}$ coisotropic with respect to $\rho(\Lambda)$.

We leave it to the reader to check that the k-vector field on $\pi_{Z(R)}$ induced in this case on the étale resolution $Z(R) = R\backslash\Gamma_L$ (provided that it exists) is simply the infinitesimal k-vector field associated to Λ through the right action of $\Gamma \rightrightarrows M$ on $Z(R)$.

6.4 The Grothendieck resolution with its Evens-Lu Poisson structure.

We show that the Grothendieck resolution is an example of resolution of a (holomorphic) algebroid leaf, and we use Poisson groupoids to turn the Grothendieck resolution in a Poisson resolution. This Poisson structure is precisely the one discovered by Sam Evens and Jiang-Hua Lu in [11].

Any Lie group G acts on itself by conjugation, so that one can form the action groupoid $G \times \underline{G} \rightrightarrows \underline{G}$ (here $\underline{G}$ stands for the Lie group G when it can be considered as a manifold acted upon by conjugation, while we keep the notation G when G is considered as a Lie group). Recall that the source map s is given by $(g,h) \to h$, the target map t is given by $(g,h) \to g^{-1}hg$ and the product is given by $(g_1,h_1)\cdot(g_2,h_2) = (g_1g_2, h_1)$ whenever $h_2 = t(g_1,h_1)$.

The Grothendieck resolution.

Assume now that G is a complex simple, connected and simply-connected Lie group. Let H be a Cartan subgroup, B a Borel subgroup containing H, and U the unipotent radical of B.

We choose some $t \in H$ and denote by $\mathcal{S}$ the regular orbit (= conjugacy class) containing t in its closure, see [14]. The closure $\overline{\mathcal{S}}$ of $\mathcal{S}$ is what is called a *Steinberg fiber* and denoted by F_t in [11].

Lemma 6.6. *The submanifold $L = tU$ is an algebroid crossing of $\overline{\mathcal{S}}$ with normalisation* $\mathrm{Lie}(R) \to tU$.

Proof. Any $g \in G$ belongs to a Borel subgroup, and all Borel subgroups are conjugate. In particular, any $g \in \overline{\mathcal{S}}$ is conjugate to an element in $t'U$, with $t' \in H$. Since the intersection of two Steinberg fibers is empty (see section 3.2 in [11]), one needs to have $t = t'$, and tU intersects all the algebroid leaves contained in $\overline{\mathcal{S}}$.

The Lie groupoid $R = B \times tU \rightrightarrows tU$ is a closed sub-Lie groupoid of the Lie groupoid $\Gamma = G \times \underline{G} \rightrightarrows \underline{G}$, and:

$$\Gamma_{L\cap\mathcal{S}}^{L\cap\mathcal{S}} \subset R. \tag{26}$$

Let $\mathrm{Lie}(R) \to tU$ be the Lie algebroid of $R \rightrightarrows tU$. For all $g \in tU \cap \mathcal{S}$, Eq. (26) amounts to the fact that $\mathrm{Lie}(R) = \rho^{-1}(T_g tU)$, so that $\mathrm{Lie}(R) \to tU$ is the normaization of tU. By construction $L \cap \mathcal{S}$ is a dense open subset of $L = tU$. This completes the proof. $\square$

By Lemma 6.6 and (26), all the assumptions of Proposition 2.11(3) are satisfied, and one can construct a resolution of $\overline{\mathcal{S}}$, that we denote by (X_t, μ) in order to follow the notations of [11] again. Let us give explicitly the construction of X_t and μ. To start with, we clearly have $\Gamma_L = G \times L = G \times tU$, while the quotient under the left action of the Lie sub-groupoid $R \rightrightarrows L$ is the quotient of $G \times tU$ through the action of the Lie group B given by

$$b \cdot (g, tu) = (bg, btub^{-1}) \quad \forall b \in B, g \in G, u \in U. \tag{27}$$

The map μ is the map $\mu([g, tu]) \to Ad_g(tu)$ (where $[g, tu] \in \frac{G \times L}{B}$ is the class modulo the action of B of the element $(g, tu) \in G \times L$).

A short comparison with Section 3.2 in [11] shows that the resolution (X_t, μ) coincides with the Grothendieck resolution (also called Springer resolution). In conclusion, we have the following proposition.

Proposition 6.7. *Let G be a simple, connected and simply-connected complex Lie group, H a Cartan subgroup and $t \in H$. Let $\mathcal{S} \subset G$ be a regular orbit with $t \in \overline{\mathcal{S}}$, and $F_t = \overline{\mathcal{S}}$ be the Steinberg fiber. Let (X_t, μ) be as above.*

The resolution (X_t, μ) is a resolution of the Steinberg fiber $F_t = \overline{\mathcal{S}}$.

The Evens-Lu Poisson structure.

Now, we endow the resolution (X_t, μ) of F_t with a Poisson structure, following [11] as a guideline again. The construction below matches step by step the construction in [11], only the interpretation claims to be new.

Let $\mathfrak{g}$ be the Lie algebra of G, and . The Lie algebra $\mathfrak{h}$ of H is a Cartan subalgebra, and we can choose a root decomposition $\Phi = \Phi^+ \cup \Phi^-$ and root vectors $E_\alpha, E_{-\alpha}, \alpha \in \Phi_+$ so that the space

$$\mathfrak{n} = \oplus_{\alpha \in \Phi^+} \mathbb{C} E_\alpha$$

is the Lie algebra of U. Also we define

$$\mathfrak{n}_- = \oplus_{\alpha \in \Phi^+} \mathbb{C} E_{-\alpha}.$$

We recall the construction of the standard Manin triple [16]. Let $\mathfrak{g}_\Delta$ be the diagonal of $\mathfrak{g} \otimes \mathfrak{g}$, and

$$\mathfrak{g}^*_{st} = \{(x + y, -y + z) \,|\, x \in \mathfrak{n}, z \in \mathfrak{n}_-, y \in \mathfrak{h}\}.$$

Then $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g}_\Delta, \mathfrak{g}^*_{st})$ is a Manin triple.

According to [15], Section 4.5 (in particular Theorem 4.21), to any Manin triple $(\mathfrak{d}, \mathfrak{g}_1, \mathfrak{g}_2)$, is associated a natural multiplicative Poisson structure on the transformation groupoid $G_1 \times D/G_1 \rightrightarrows D/G_1$, where D and $G_1 \subset D$ are connected and simply-connected Lie groups integrating $\mathfrak{d}$ and $\mathfrak{g}_1$ respectively, and where G_1 acts on D/G_1 by left multiplication. In the present case, $D = G \times G$ and G_1 is the diagonal G_Δ of $G \times G$, so that D/G_1 can be identified with G by mapping $[g_1, g_2] \in D$ to $g_1 g_2^{-1} \in G$, where $[g_1, g_2]$ stands for the class of (g_1, g_2) modulo the action of B given by Equation (27). Under this isomorphism the G-action on D/G becomes the conjugation of G on $\underline{G}$, so that the Lie groupoid $G \times D/G \rightrightarrows D/G$ can be identified with the Lie groupoid $G \times \underline{G} \rightrightarrows \underline{G}$ previously described. In conclusion, the Lie groupoid $G \times \underline{G} \rightrightarrows \underline{G}$ can be endowed with a Poisson structure $\pi_{G \times \underline{G}}$ that turns it into

a Poisson groupoid (i.e. a Lie groupoid endowed with a multiplicative Poisson bivector field, see [27]). A precise description of $\pi_{G\times\underline{G}}$ is given by Equation (77) in [15].

Since the pair $(G\times\underline{G}\rightrightarrows\underline{G},\pi_{G\times\underline{G}})$ is a Poisson Lie group, there exists a Poisson structure $\pi_{\underline{G}}$ on the base manifold $\underline{G}$ such that $\pi_{\underline{G}}=s_*(\pi_{G\times\underline{G}})$ as in Equation (3).

We wish to compare these vector fields with the Poisson structure on $G\times G$ called π_D^+ and the Poisson structure on G called π introduced in [11], Section 2.2:

Lemma 6.8. *1. The Poisson structures π and $-\pi_{\underline{G}}$ coincide.*

2. The Poisson structure $\pi_{G\times\underline{G}}$ is the image of the Poisson structure $-\pi_D^+$ through the map $(g_1,g_2)\rightarrow(g_2,g_1g_2^{-1})$

Proof. We only prove the first point, since it is the only point really needed to prove Proposition 6.10. The second one is just a cumbersome computation.

According to Proposition 4.19 in [15], the Poisson structure induced on $\underline{G}\simeq G\times G/G_\Delta$ (which is the space denoted by S in [15]) is equal to:

$$\pi_{\underline{G}}=-\sum_{i=1}^{d}(e_i)_S\wedge(\varepsilon_i)_S \tag{28}$$

where X_S stands for the infinitesimal vector field induced by the action of an element $X\in\mathfrak{d}=\mathfrak{g}\oplus\mathfrak{g}$ on $\underline{G}\simeq G\times G/G_\Delta$, and where $(e_i)_{i=1}^d,(\varepsilon_i)_{i=1}^d$ are dual bases of $\mathfrak{g}_{st}^*$ and $\mathfrak{g}_\Delta$ respectively. But, under the identification $G\times G/G_\Delta\simeq G$ mapping $[g_1,g_2]$ to $g_1^{-1}g_2$, as previously described, we have $(X,Y)_S=X^L-Y^R$, where X^L and Y^R are the left and right actions of $X,Y\in\mathfrak{g}$ on G respectively. The construction of $\pi_{\underline{G}}$ given in Eq. (28) coincides then with the construction of π described in Equation (2.10) in [11], up to a sign. □

Remark 6.9. To find again the explicit form of π given in Equation (2.10) in [11], one can choose the dual bases of $\mathfrak{g}_{st}^*$ and $\mathfrak{g}_\Delta$ given by

$$\left\{(y_1,-y_1),\ldots,(y_r,-y_r),(0,-E_{-\alpha}),(E_\alpha,0)\,|\alpha\in\Phi^+\right\}$$

and

$$\left\{(y_1,y_1),\ldots,(y_r,y_r),(E_\alpha,E_\alpha),(E_{-\alpha},E_{-\alpha})\,|\alpha\in\Phi^+\right\}.$$

where $(y_i)_{i=1}^r$ is a base of $\mathfrak{h}$ with $2\langle y_i,y_j\rangle=\delta_i^j$, and where we assume that $\langle E_\alpha,E_{-\alpha}\rangle=1$ for all $\alpha\in\Phi^+$. In the previous, $\langle\cdot,\cdot\rangle$ stands of course for the Killing form.

According to Lemma 3.7 in [11], the submanifold $L=tU$ is coisotropic with respect to $\pi_{\underline{G}}$ (which, according to Lemma 6.8(1), coincides with the Poisson structure denoted by π is [11]). All the conditions of Theorem 3.7(4) are therefore satisfied. We can construct a Poisson structure on $X_t=\frac{G\times L}{B}$ such that $\mu:X_t=\frac{G\times L}{B}\rightarrow\underline{G}$ is a Poisson map, id. est. (X_t,μ) is a resolution compatible with the Poisson structure $\pi_{\underline{G}}$. In conclusion, we have proved the follwing proposition:

Proposition 6.10. *Let G be a simple connected and simply-connected complex Lie group, and H a Cartan subgroup. Let $\mathcal{S}\subset G$ be a regular orbit with $t\in\overline{\mathcal{S}}$.*

The Poisson structure $\pi_{\underline{G}}$ is tangent to the Steinberg fiber $F_t=\overline{\mathcal{S}}$, and the resolution (X_t,μ) of F_t is a Poisson resolution.

Since $\mu^{-1}(\mathcal{S})$ is open and dense in X_t, the Poisson structure on X_t such that μ is a Poisson map is unique, and the one constructed here needs to coincide with the one constructed in [11]. Proposition 6.10 therefore reproves Proposition 4.5 (2) is [11]. Indeed, a step by step comparison, with the help of Lemma 6.8(2), shows that the present construction of the Poisson structure on X_t as exposed in the proof of Theorem 3.7(4), coincides precisely with the construction described in [11].

6.5 Minimal resolutions of $\mathbb{C}^2/(\mathbb{Z}/l\mathbb{Z})$.

Let $l \in \mathbb{N}^*$ be an integer. The group $\mathbb{Z}_l = \mathbb{Z}/l\mathbb{Z}$, seen as the group of l^{th} roots of the unity, acts on $\mathbb{C}^2$ by $\lambda \cdot (z_1, z_2) = (\lambda z_1, \overline{\lambda} z_2)$, for all $\lambda \in \mathbb{Z}_l$. The quotient space $W_l = \mathbb{C}^2/\mathbb{Z}_l$ is an affine variety that can be described as the zero locus in $\mathbb{C}^3$ of the function

$$\chi_l(x, y, z) := xy - z^l.$$

The variety W_l has only one singular point O; when seen as a subvariety of $\mathbb{C}^3$, this singular point is the origin.

Let us recall several facts about its canonical Poisson structure. The action of G preserves the canonical symplectic structure on $\mathbb{C}^2$, hence the canonical Poisson bracket on $\mathbb{C}^2$ goes to the quotient and induces a Poisson bracket $\{\cdot,\cdot\}_{W_l}$ on W_l. This Poisson structure π_{W_l} is symplectic at all regular points of W_l.

Alternatively, it can be described as follows. Consider the following Poisson bracket $\pi_{\mathbb{C}^3} = \{\cdot,\cdot\}_{\mathbb{C}^3}$ on $\mathbb{C}^3$:

$$\{x, y\}_{\mathbb{C}^3} = \frac{\partial \chi_l}{\partial z},\ \{y, z\}_{\mathbb{C}^3} = \frac{\partial \chi_l}{\partial x},\ \{z, x\}_{\mathbb{C}^3} = \frac{\partial \chi_l}{\partial y}. \tag{29}$$

Then χ_l is a Casimir function of the latest bracket, so that the Poisson structure it defines induces a Poisson structure on the zero locus W_l of χ_l.

Lemma 6.11. *There exists a algebraic Poisson variety (N, π_N) such that*

1. *N is a nonsingular variety.*
2. *(N, π_N) is integrable (when seen as a holomorphic Poisson manifold).*
3. *(N, π_N) admits a symplectic leaf $\mathcal{S}$ whose closure $\overline{\mathcal{S}}$ is a subvariety of N isomorphic to W_l as an algebraic Poisson manifold. moreover, $\mathcal{S}$ is the regular part of $\overline{\mathcal{S}}$.*

Proof. A natural candidate for (N, π_N) would be $(\mathbb{C}^3, \pi_{\mathbb{C}^3})$. But it is not clear that this structure is integrable. But $(\mathbb{C}^3, \pi_{\mathbb{C}^3})$ is, according to Theorem 5.5 in [5], a Poisson submanifold of a Poisson manifold N (and also denoted N in [5]) that we now describe.

Let $e \in \mathrm{sl}_l(\mathbb{C})$ be an element of the subregular nilpotent orbit, and $\mathfrak{n} \subset \mathrm{sl}_l(\mathbb{C})$ be a complement of the centraliser of e. Then the affine space $N := x + \mathfrak{n}^{\perp}$ is of course a nonsingular submanifold, so that condition 1) is satisfied.

According to [21], one can choose $\mathfrak{n}$ so that the Poisson matrix of the linear Poisson structure of $\mathrm{sl}_l(\mathbb{C})$ is, for all $y \in N$, of the form

$$\begin{pmatrix} A(y) & B(y) \\ -B^{\perp}(y) & C(y) \end{pmatrix} \tag{30}$$

where $C(y)$ is an invertible matrix (more precisely, a matrix of determinant 1, see the proof of Theorem 2.3 in [21]). This amounts to the fact that N is a Dirac submanifold of $\mathrm{sl}_l(\mathbb{C})$. The classical procedure called Dirac reduction in [21] (and sometimes called Poisson-Dirac reduction) yields then a Poisson structure π_N on N, which is polynomial by construction. In conclusion (N, π_N) is an algebraic Poisson variety.

Let us show that it is integrable. First, $\mathrm{sl}_l(\mathbb{C})$ is an integrable Poisson manifold, and the symplectic Lie groupoid that integrates this Poisson manifold is $\Gamma = \mathrm{SL}_l(\mathbb{C}) \times \mathrm{sl}_l(\mathbb{C}) \rightrightarrows \mathrm{sl}_l(\mathbb{C})$. Since the matrix $C(y)$ is invertible for all $y \in N$, N is a cosymplectic submanifold of $\mathrm{sl}_l(\mathbb{C})$ (see [4], a cosymplectic manifold is a Poisson-Dirac manifold with a symplectic transverse structure). This implies that $\Gamma_N^N \rightrightarrows N$ is a symplectic sub-Lie groupoid of $\Gamma \rightrightarrows \mathrm{sl}_l(\mathbb{C})$, and the Poisson structure it integrates is the Dirac-Poisson structure on N, see [4]. Hence N is a Poisson-Dirac submanifold. $\square$

For all $k \in \{1, \cdots, l-1\}$, the subvariety of $\mathbb{C}^3$

$$L_k := \{(\lambda^k, \lambda^{l-k}, \lambda), \lambda \in \mathbb{C}\}$$

is a subvariety of W_l. It can also be described by the equations:

$$x = z^k \text{ and } y = z^{l-k}. \tag{31}$$

Lemma 6.12. *For all $k \in \{1, \cdots, n-1\}$, the subvariety L_k is a Lagrangian crossing of $\overline{\mathcal{S}}$.*

Proof. The subvariety L_k is nonsingular, since it is given by (31) when seen as a subvariety of $\mathbb{C}^3$ via the inclusion $W_l \subset \mathbb{C}^3$ described above. The intersection of L_k with $\mathcal{S}$ is of course Lagrangian since it has dimension 1. Now, the identity $\overline{\mathcal{S}} = \mathcal{S} \cup \{O\}$ holds, the subvariety L_k contains the point O, and has a nonempty intersection with $\mathcal{S}$. All the assumptions are therefore satisfied. $\square$

The variety W_l admits a minimal resolution (Σ, ϕ), see [22] Section IV-4-3, and it is well-known that this resolution is symplectic, see Example 2.2 in [9].

Let us say a few words on this resolution.

Explicitly, (Σ, ϕ) is constructed with a help of $l-1$ successive blowup, as follows. The blowup of W_l at 0 is the projective variety of $P^3(\mathbb{C})$ given, in the three canonical charts, by the equations

$$x = z^l y^{l-2}, \quad y = z^l x^{l-2}, \quad \psi_{l-2}(x, y, z) = 0$$

with the usual gluing relations. It is easy to check that the inverse image of 0 consists of two copies of $P^1(\mathbb{C})$ that intersect transversally at a point if $l \neq 2$, and consists of one copy of $P^1(\mathbb{C})$ if $l = 2$. Also, the two first components of the blowup $\tilde{W}$ are nonsingular, while the last one is isomorphic to W_{l-2}, to which the procedure can be applied recursively until the nonsingul ar varieties W_0 or W_1 appear. Applying successive blowup therefore, one gets a resolution (Σ, ϕ) of W_l and a closed look at the construction amounts to the following properties.

Lemma 6.13. *1. (Σ, ϕ) is the minimal resolution.*

2. (Σ, ϕ) *is a proper symplectic resolution which is compatible with the Lagrangian crossing* L_k *for all* $k = 1, \dots, l-1$. *Let* $\tilde{L_k}$ *be the submanifold of* Σ *to which the restriction of* $\tilde{L_k}$ *is a biholomorphism onto* L_k.

3. *The inverse image of* 0 *consists of* $l-1$ *projective curves* $C_1, \cdots, C_{l-1}$, *all isomorphic to* $\mathbb{P}^1(\mathbb{C})$.

4. *For all* $k = 1, \cdots, l-1$, $\tilde{L_k}$ *intersects* C_k *transversally at exactly one point, and* $\tilde{L_k} \cap C_i = \emptyset$ *for* $k \neq i$.

5. *For all* $k = 1, \cdots, l-2$, C_k *intersects* C_{k+1} *at exactly one point* p_k, *and* $C_j \cap C_i = \emptyset$ *for* $|j - i| \geq 2$.

6. *The kernel of the differential of* ϕ *at a point* $y \in C_k$ *distinct from* p_k *or* p_{k-1} *is equal to* $T_y C_k$.

7. *The differential of* ϕ *vanishes at the points* $p_1, \dots, p_{l-2}$.

Let $\Gamma \rightrightarrows M$ be the source-simply connected symplectic Lie groupoid that integrates (M, π_M). All the assumptions of Theorem 5.5 are satisfied, so that there exists, for all $k = 1, \dots, l-1$, a closed sub-Lie groupoid $R_k \rightrightarrows L_k$ of $\Gamma \rightrightarrows M$ that integrates $TL_k^{\perp} \to L_k$ and contains $\Gamma_{L_k \cap \mathcal{S}}^{L_k \cap \mathcal{S}}$, and there exists an open subset U_k of Σ isomorphic to the symplectic resolution associated to $R_k \rightrightarrows L_k$ as in Theorem 4.6 (3). The following Proposition describes in a very explicit way this open subset:

Proposition 6.14. *For all* $k = 1, \cdots, l-1$, *we have*

$$U_k = \Sigma - \cup_{i \neq k} C_i$$

Proof. We assume $k \neq 1$ and $k \neq l-1$ for simplicity. The cases $k = 1$ and $k = l-1$ can be dealt in the same way. Let $O_k \in \tilde{L_k}$ be the inverse image of O through the restriction of ϕ to a biholomorphism from $\tilde{L_k}$ onto L_k. Notice that O_k belongs to C_k, according to Lemma 6.13(4).

By construction, U_k is equal to

$$\Gamma \cdot (\tilde{L_k} \backslash \{O_k\}) \cup \Gamma \cdot \{O_k\},$$

where $\Gamma \cdot \{\tilde{L_k} \backslash O_k\}$ (resp. $\Gamma \cdot \{O_k\}$) stands for the Γ-orbit of $\tilde{L_k} \backslash \{O_k\}$ (resp. O_k) with respect to the action of $\Gamma \rightrightarrows M$ on Σ defined in Proposition 5.6. Proposition 5.6 gives the identity $\Gamma \cdot (\tilde{L_k} \backslash \{O_k\}) = \phi^{-1}(\mathcal{S})$, since the inclusion $\tilde{L_k} \backslash \{O_k\} \subset \phi^{-1}(\mathcal{S})$ holds by Lemma 6.13(4). We therefore have

$$U_k = \phi^{-1}(\mathcal{S}) \cup \Gamma \cdot \{O_k\}.$$

Now, Proposition 5.6, together with Equation (20) give that a point $y \in \phi^{-1}(O)$ belongs to U_k if and only if there exists a path $a(t)$ in $T_O^* M$ together with a smooth path $\sigma(t)$ in Σ (with $t \in [0, 1]$) such that, first, $\sigma(1) = x$ and $\sigma(0) = O_k$, and second:

$$\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t} = (\pi_\Sigma^{\#})_{|\sigma(t)} \big(a(t) \circ \mathrm{d}_{\sigma(t)} \phi\big). \tag{32}$$

Assume that $y \in \cup_{i \neq k} C_i$, then $\sigma(t)$ has to go through one at least of the points p_{k-1} or p_k for some $t = t_0$. But then, there can not exist a smooth path $a(t)$ that satisfies (32) since

the right hand side of (32) would vanish at $t = t_0$ by Lemma 6.13(7), so that $\sigma(t)$ would have to be a constant path. Hence, y has to be a point in C_k.

For all $y \in C_k$, there exists a smooth path $\sigma(t)$, taking values in $C_k \backslash \{p_k, p_{k+1}\}$, such that $\sigma(0) = O_k, \sigma(1) = y$. Since the image of the dual map of $\mathrm{d}_{\sigma(t)}\phi$ has rank 1 for all t by Lemma 6.13(6), it contains the covector $\omega_\sigma(\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t}, \cdot)$. There exists therefore a path $a(t) \in T^*_O M$ which satisfies Eq. (32), and $x \in \Gamma \cdot O_k$. this completes the proof. □

Remark 6.15. W_l is a particular type of Kleinian singularity. It is natural to ask whether these constructions could be done for other Kleinian singularities, those of type $D_l, l \geq 4$, or E_6, E_7, E_8. The answer is negative in general, due to the lack of Lagrangian crossing in these cases.

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